Cohomology of combinatorial species

Pedro Tamaroff
Universidad de Buenos Aires

August 2017
Goals

- Study the category $\mathbb{k}\text{Sp}$ of combinatorial species with its monoidal structure given by the *Cauchy product*.

- Study its coalgebras, in particular the *exponential species* $e$: a coalgebra in $\mathbb{k}\text{Sp}$ is the same as an $e$-bicomodule.

- We get a functor

\[ x \text{ coalgebra in } \mathbb{k}\text{Sp} \mapsto H^*(x) = \text{Ext}^*_{e-e}(x, e) \]

which we want to understand as completely as possible.
Combinatorial species

Fix a commutative ring $\mathbb{k}$. A *combinatorial species over $\mathbb{k}$* is a functor

$$x : \text{FinSet}^\times \longrightarrow \mathbb{k}\text{Mod}.$$  

Such functors are the objects of a $\mathbb{k}$-linear abelian category $\mathbb{k}\text{Sp}$.

If $x$ and $y$ are species, their Cauchy product is such that

$$(x \otimes y)(I) = \bigoplus_{S \sqcup T = I} x(S) \otimes_{\mathbb{k}} y(T)$$

Along with more data, this endows the category $\mathbb{k}\text{Sp}$ with a symmetric monoidal structure.
The upshot of a monoidal structure

Coalgebras, algebras and Hopf algebras in $\mathbb{k}\text{Sp}$ codify useful and classical combinatorial operations of “restriction” and “gluing” of structures.

**Example:** if $P$ is the species of posets, there is $\mu : P \otimes P \rightarrow P$ such that

$$\mu(p^1 \otimes p^2) = p^1 \ast p^2 \quad \text{(ordinal sum)}$$

**Example:** if $G$ is the species graphs, there is $\Delta : G \rightarrow G \otimes G$ such that

$$\Delta(g) = \sum gs \otimes gT \quad \text{(induced subgraphs)}$$
The exponential species $e : \text{Set}^\times \to \mathbb{K}\text{Sp}$ is such that $e(I) = \mathbb{K}\{e_I\}$ for each finite set $I$. It is a bialgebra, with

$$e_\mathcal{S} e_\mathcal{T} = e_{\mathcal{S} \cup \mathcal{T}}, \quad \Delta(e_I) = \sum e_\mathcal{S} \otimes e_\mathcal{T}$$

The category of “linear” $e$-bicomodules is equivalent to that of “linear” coalgebras.
To each linear coalgebra $x$ in $\mathbb{k}Sp$ we associate cohomology groups

$$H^*(x) = \text{Ext}_{e-e}^*(x, e).$$

We can compute $H^*(x)$ with a canonical complex that resolves $e$ by applying $\text{hom}_{e-e}(x, ?)$ to

$$\Omega e : 0 \rightarrow e^\otimes 2 \rightarrow e^\otimes 3 \rightarrow e^\otimes 4 \rightarrow \ldots$$

**Problem:** although theoretically useful, this complex is computationally ineffective.
To each linear coalgebra $\mathbf{x}$ in $\mathbb{k}\text{Sp}$ we associate cohomology groups

$$H^*(\mathbf{x}) = \text{Ext}^*_{\mathbb{k}e}(\mathbf{x}, \mathbb{k}e).$$

We can compute $H^*(\mathbf{x})$ with a canonical complex that resolves $\mathbb{k}e$ by applying $\text{hom}_{\mathbb{k}e}(\mathbf{x}, ?)$ to

$$\Omega \mathbb{k}e : 0 \rightarrow \mathbb{k}e \otimes \mathbb{k}e \rightarrow \mathbb{k}e \otimes \mathbb{k}e^2 \rightarrow \mathbb{k}e \otimes \mathbb{k}e^3 \rightarrow \mathbb{k}e \otimes \mathbb{k}e^4 \rightarrow \cdots$$

**Problem**: although theoretically useful, this complex is computationally ineffective.
Let $L : \text{Set}^\times \to \mathbb{k}\text{Mod}$ be the species of linear orders. This is the free noncommutative algebra on one generator on cardinality 1.

**Theorem**

If $\mathbb{k}$ is a PID and if $\mathbb{Q} \subseteq \text{Frac}(\mathbb{k})$, there is an algebra isomorphism

$$H^*(L) \simeq \mathbb{k}[\xi, \eta]$$

with $|\eta| = 1$ and $|\xi| = 2$.

In particular $H^n(L)$ is one dimensional for each $n \in \mathbb{N}_0$. 
Concrete computations: simplicial complexes

We can associate to every finite simplicial complex $K$ a left $e$-comodule $x_K$.

**Theorem**

If $k$ is a PID, there are isomorphisms

$$H^*(x_K^t) \cong H^*(K)[-1]$$

$$H^*(x_K^s) \cong SR(K)$$

where $SR(K)$ is the graded Stanley-Reisner ring associated to $K$.

**Example:** if $K = \partial \Delta^2$, then

$$SR(K) = \Lambda[x, y, z]/(xyz)$$
The Coxeter complex and the braid arrangement

**Key step:** use the explicit description of the $S$-module structure of the cohomology groups of the Coxeter complexes of type $A$. Their complexes of simplicial chains appear naturally as the sequence

\[
\left\{ 0 \longrightarrow e \otimes^0(n) \longrightarrow e \otimes^1(n) \longrightarrow e \otimes^2(n) \longrightarrow \cdots \right\}_{n \geq 0}
\]
We were able to abstract a general method from our computations.

**Theorem 1**

Let \( x \) be an \( e \)-bicomodule. There is a cochain complex \( CC^*(x) \) with

\[
CC^n(x) = \text{hom}_{S_n}(x(n), k(n)^-)\]

so that

\[
H(CC^*(x)) \cong H^*(x)
\]

if \( x(n) \) is \( k[S_n] \)-projective for each \( n \in \mathbb{N}_0 \).

Moreover, if \( x \) is a coalgebra, \( CC^*(x) \) is a DGA algebra, and the isomorphism is one of DGA algebras.
The combinatorial complex

We were able to abstract a general method from our computations.

**Theorem 1**

Let $\mathbf{x}$ be an $\mathbf{e}$-bicomodule. There is a cochain complex $CC^*(\mathbf{x})$ with

$$CC^n(\mathbf{x}) = \text{hom}_{S_n}(\mathbf{x}(n), \mathbb{k}(n)^-)$$

so that

$$H(CC^*(\mathbf{x})) \cong H^*(\mathbf{x})$$

if $\mathbf{x}(n)$ is $\mathbb{k}[S_n]$-projective for each $n \in \mathbb{N}_0$.

Moreover, if $\mathbf{x}$ is a coalgebra, $CC^*(\mathbf{x})$ is a DGA algebra, and the isomorphism is one of DGA algebras.
With the combinatorial complex we were able to

1. Give an alternative description of $H^*(x)$.
2. Provide a computationally simple formula for the product in $H^*(x)$.
3. Deduce structural results:
   - a Künneth formula,
   - graded commutativity for commutative coalgebras,
   - vanishing of cohomology groups
Example:

- Theorem I gives that $CC^n(L) \cong k$ for each $n \in \mathbb{N}_0$.
- As $L$ is commutative, by Theorem II we have $d = 0$.
- Thus $H^n(L) \cong k$ for each $n \in \mathbb{N}_0$, which recovers our computation.

More generally: if $S_x = e \circ x$ is the free commutative coalgebra on a positive species $x$, $d = 0$ and the combinatorial complex is the cohomology algebra of $S_x$. 
The differential

**Theorem II**

If \( \mathbb{k} \) is a field of characteristic zero, the differential \( d \) of \( CC^*(x) \) is such that, for each \( f \in CC^n(x) \) and \( z \in x(n+1) \),

\[
d(f)(z) = \sum_{i=1}^{n+1} (-1)^{i-1} (f(z'_i) - f(z''_i)).
\]

In particular, \( d = 0 \) if \( x \) is a cosymmetric bicomodule.
Some examples

**Graphs:** we can endow the species of graphs $G$ with a non-commutative comultiplication. With this structure:

\[
\begin{pmatrix}
2 & 5 \\
1 & 4 \\
3 & 6 \\
\end{pmatrix}
\begin{pmatrix}
2 & 5 \\
1 & 4 \\
3 & 6 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 & 5 \\
3 & 6 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 & 5 \\
3 & 6 \\
\end{pmatrix}
\]
The spectral sequence

The argument that gives rise to $CC^*(x)$ can be generalized to (attempt to) compute $\text{Ext}^*_h(x, h)$ with $h$ a coalgebra in $\mathbb{k} \text{Sp}$.

**Theorem IV**

There is a spectral sequence with

$$E_1^{p,q} \simeq \text{hom}_{S_p}(x(p), H^{p,q}(h)) \Rightarrow \text{Ext}^{p+q}_{h-h}(x, h),$$

where $H^{p,q}(h) = \text{Cotor}_{h}^{p,p-q}(\mathbb{k}, \mathbb{k})$.
Computing these Cotor groups is a complex combinatorial problem —this includes understanding them as $\mathbb{S}$-modules! We did this for $e$, where we know

$$\text{Cotor}_{e}^{p,q}(k, k) = \begin{cases} k^{-}(p) & \text{if } p = q, \\ 0 & \text{else.} \end{cases}$$

**Problem:** for the species $L$ of linear orders, with the aid of a computer, we obtained that for $0 \leq p, q \leq 5$

$$\dim_{k} \text{Cotor}_{L}^{p,q}(k, k) = s(p, q),$$

where the numbers $s(n, k)$ are the Stirling numbers of the first kind. We can prove this for entries $(p, p)$, where we know $\text{Cotor}_{L}^{p,p}(k, k)$ is the sign representation of $S_p$. What about the others?
Questions?


