The non-commutative calculus of fields and forms through dg-resolutions

Pedro Tamaroff
Trinity College Dublin

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Motivation and origins: the Cartan calculus

For a smooth manifold $M$, the spaces $\Omega(M)$ of forms on $M$ and $\Theta(M)$ of polyvector fields on $M$ are endowed with a Cartan calculus.

Similarly, for a smooth commutative algebra $A$, we know from the HKR theorem that we have identifications

$$\text{HH}_*(A) = \Lambda^*_A \Omega^1_A, \quad \text{HH}^*(A) = \Lambda^*_A \text{Der}(A).$$

which give us a “Cartan calculus” for $A$: a wedge product on fields, a contraction of forms with fields, a de Rham differential on forms, and a Lie bracket on fields.
The non-commutative analogue

We can produce an analogous picture when $A$ is an arbitrary associative algebra (Daletski–Gelfand–Tsygan ’90), the Tamarkin–Tsygan calculus of $A$, and write it

$$\text{Calc}(A) = (\text{HH}^*(A), \text{HH}_*(A)).$$

This is a pair of the form $(V, M)$ where $V$ is a Gerstenhaber algebra and $M$ is a $V$-module along with a differential $d$ relating the Lie module and the module structure of $M$ through “Cartan’s magic formula”:

$$[i, d] = L.$$

**Theorem.** (Armenta–Keller ’18) The Tamarkin–Tsygan calculus of an algebra is derived invariant.
An definition intrinsic to dg resolutions

The above produces an assignment (not a functor) from associative algebras to Tamarkin–Tsygan calculi.

From the work of Jim Stasheff ('93), we know the bracket is “intrinsic” to the homotopy category of dg algebras: we can compute it as the Lie bracket on derivations of any good dg resolution of our algebra.

**Question.** What about the whole Tamarkin–Tsygan calculus? Can we produce from the homotopy type of $A$ a datum that gives this calculus and from which it can be effectively computed?

From now on, let us fix a dg replacement $(TV, \partial) = B \to A$. 
Standard resolution

If $TV = B$ is a free algebra, there is a “standard” resolution in $B\text{Mod}_B$

$$\text{St}_*(B): 0 \rightarrow B \otimes V \otimes B \rightarrow B \otimes B \rightarrow B \rightarrow 0$$

where, in addition, we have internal differentials coming from $\partial$.

If $\text{Bar}_*(B)$ is the double-sided bar resolution, there is a retraction of resolutions

$$\pi : \text{Bar}_*(B) \rightarrow \text{St}_*(B), \quad i : \text{St}_*(B) \rightarrow \text{Bar}_*(B).$$

where $i$ is the inclusion and $\pi$ is very simple.

**Conclusion:** we can compute the underlying (co)homology groups of $\text{Calc}(A)$ through the standard resolution $\text{St}_*(B)$. 
Non-commutative fields and forms

Note that the complexes $\text{St}_*(B)_B$ and $\text{St}_*(B)^B$ are in fact naturally isomorphic to

$$\mathcal{V}(B) = (\text{ad} : B \to \text{hom}(V, B)), \quad \Omega(B) = (\text{co} : B \otimes V \to B)$$

respectively, which we call the complexes of non-commutative fields and non-commutative forms on $B$.

**Problem:** we can compute the calculus of $A$ through $\text{Bar}_*(B)$, but can we do this with these smaller complexes?

**Answer:** this depends on how well we understand how calculi behave under retractions!
A structure on Hochschild (co)chains

Deligne’s question: can one lift the Gerstenhaber algebra structure on \( \text{HH}^*(A) \) to the chain level? Yes, the solution involves formality of the little disks operad.

It is reasonable to consider the same problem for the Tamarkin–Tsygan calculus structure on \( \text{Calc}(A) \).

**Theorem** (Kontsevich-Soibelman) There is a formal geometric operad \( C \) that solves Deligne’s conjecture for \( \text{Calc}(A) \): there is an action of \( C \) on the pair \( (C^*(A), C_*(A)) \) so that taking homology we get the usual calculus.
Homotopy calculi

- Classical structures (commutative, Lie, associative, Gerstenhaber) have “homotopy coherent” versions.
- One can do the same for calculi if one finds a dg replacement of the operad Calc controlling calculi.
- Note this operad admits a quadratic-cubic presentation, owing to the Cartan magic formula.

**Theorem** (T.) The operad Calc is inhomogeneous Koszul.

It follows that one can consider a reasonable notion of homotopy coherent calculi, and this notion behaves just as good as the classical ones.
Homotopy transfer

To solve our problem above, we put together
- the result of Kontsevich–Soibelman and
- the dg replacement $\text{Calc}_\infty$ of $\text{Calc}$.

**Corollary** (Daletskii–Tamarkin–Tsygan) For every algebra $A$, the pair of Hochschild cochains $(C^*(A), C_*(A))$ admits a homotopy coherent calculus structure.

**Corollary** (T.) The pair $(\mathcal{V}(B), \Omega(B))$ admits a homotopy coherent calculus structure that is equivalent to the homotopy coherent calculus on $(C^*(A), C_*(A))$. 
A small quiver

Let us consider the following quiver $Q$ with relations $R = \{xy^2, y^2z\}$. We will compute its minimal dg resolution and with part of its calculus.

The dg replacement $B$ is given by the free algebra over $\mathbb{k}Q_0$ with set of homogeneous generators $\{x, y, z, \alpha, \beta, \Gamma, \Lambda\}$ such that

\[
\partial x = \partial y = \partial z = 0, \\
\partial \alpha = xy^2, \quad \partial \beta = y^2z, \\
\partial \Gamma = \alpha z - x\beta, \quad \partial \Lambda = xy\beta - \alpha yz.
\]
The (dg) quiver of $B$ looks as follows

and we now consider the complex of nc fields $\mathcal{V}(B) = (B \to \text{Der}(B))$ on $B$ to compute $\text{HH}^*(A)$ (one can compute all the calculus with it!).
Computation of $\text{HH}^1(A)$

We can compute the 0-cycles directly:

\[
\begin{align*}
E_s(x) &= 0, & E_s(y) &= y^{s+1}, & E_s(z) &= 0, & E_s(\alpha) &= 2\alpha y^s, \\
E_s(\beta) &= 2y^s\beta, & E_s(\Lambda) &= 3\alpha y^{s-1}\beta, & E_s(\Gamma) &= -2\alpha y^{s-2}\beta, \\
F_s(x) &= xy^s, & F_s(y) &= 0, & F_s(z) &= 0, & F_s(\alpha) &= \alpha y^s, \\
F_s(\beta) &= 0, & F_s(\Lambda) &= \alpha y^{s-1}\beta, & F_s(\Gamma) &= -\alpha y^{s-2}\beta, \\
G_s(x) &= 0, & G_s(y) &= 0, & G_s(z) &= y^s z, & G_s(\alpha) &= 0, \\
G_s(\beta) &= y^s\beta, & G_s(\Lambda) &= \alpha y^{s-1}\beta, & G_s(\Gamma) &= -\alpha y^{s-2}\beta.
\end{align*}
\]

$\text{HH}^1(A)$ is infinite dimensional with basis the classes of the elements in $\{F_0, G_0, E_s : n \in \mathbb{N}_0\}$. For each $s, t \in \mathbb{N}_0$,

\[
[E_s, E_t] = (s - t)E_{s+t}, \quad [F_0, -] = [G_0, -] = 0.
\]

We get abelian algebra $\mathbb{k}^2$ acting trivially on the Witt algebra.
Computation of $\text{HH}^2(A)$

The following derivations form a basis of the 1-cycles in $\text{Der}(B)$, where unspecified values are zero, $s \in \mathbb{N}_0$, and we agree that $y^{-1} = y^{-2} = 0$:

\[
\begin{align*}
\Phi_s(\alpha) &= xy^s, & \Phi_s(\beta) &= y^sz, & \Phi_s(\Lambda) &= \alpha y^{s-1}z, & \Phi_s(\Gamma) &= -\alpha y^{s-2}z, \\
\Phi'_s(\alpha) &= 0, & \Phi'_s(\beta) &= y^{s+2}, & \Phi'_s(\Lambda) &= -\alpha y^{s+1}z, & \Phi'_s(\Gamma) &= \alpha y^s, \\
\Pi_s(\alpha) &= 0, & \Pi_s(\beta) &= y^{s+2}, & \Pi_s(\Lambda) &= \alpha y^{s+1}, & \Pi_s(\Gamma) &= \alpha y^s, \\
\Pi'_s(\alpha) &= xy^sz, & \Pi'_s(\beta) &= 0, & \Pi'_s(\Lambda) &= 0, & \Pi'_s(\Gamma) &= 0, \\
\Psi_s(\alpha) &= 0, & \Psi_s(\beta) &= y^{s+2}z, & \Psi_s(\Lambda) &= -\alpha y^{s+1}z, & \Psi_s(\Gamma) &= xy^s\beta, \\
\Theta_s(\alpha) &= 0, & \Theta_s(\beta) &= 0, & \Theta_s(\Lambda) &= \alpha y^sz - xy^s\beta, & \Theta_s(\Gamma) &= 0, \\
\Xi_s(\alpha) &= 0, & \Xi_s(\beta) &= 0, & \Xi_s(\Lambda) &= 0, & \Xi_s(\Gamma) &= \Theta_s(\Lambda).
\end{align*}
\]

It turns out a basis of $H^1(\text{Der}(B))$ is given by the classes of the derivations $\Phi_0, \Phi_1$ so that $\text{HH}^2(A)$ is two dimensional.
Computation of $\text{HH}^3(A)$ and the bracket

A basis for the 2-cycles is given by the following family of derivations, where $s \in \mathbb{N}_0$ and $t \in \{0, 1\}$:

$$\Omega_s^t(\Lambda) = 0, \quad \Omega_s^t(\Gamma) = xy^sz^t, \quad \gamma_s^t(\Lambda) = xy^sz^t, \quad \gamma_s^t(\Gamma) = 0.$$  

It is straightforward to check that all of these are boundaries except for $\gamma_1^0$ and $\gamma_0^0$. The bracket is as follows:

$$[E_s+2, \Phi_t] = 3\Xi_{s+t+1} - 2\Theta_{s+t},$$

$$[G_s+2, \Phi_t] = \Theta_{t+s+2} - \Xi_{t+s+2},$$

$$[E_s, \gamma_t^r] = (t - 3\delta_{s,0})\gamma_{s+t}^r.$$  

$$[F_0, -] = [G_0, -] = 2$$

$$[F_0, -] = [G_0, -] = [E_0, -] = 0$$

$$[E_1, \Phi_t] = 3\Xi_t.$$  

$$[F_{s+2}, \Phi_t] = \Theta_{t+s+1} - \Xi_{t+s},$$

$$[F_1, \Phi_t] = [G_1, \Phi_t] = \Theta_t,$$

$$[E_s, \Omega_t^r] = (t + 2\delta_{s,0})\Omega_{s+t}^r,$$

on $\langle \Omega_s^t, \Omega_{s}^t : s \in \mathbb{N}_0 \rangle$,

$$[E_s, \gamma_t^r] = (t + 2\delta_{s,0})\gamma_{s+t}^r,$$

on $\langle \gamma_s^t, \gamma_{s}^t, \Phi_s : s \in \mathbb{N}_0 \rangle$. 


Thank you!

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