

Representation Theory — WiSe 2022

Bonus exercises for Tutorial 1

1. Molien's theorem. Let V be a complex representation of a finite group G , and consider for each $g \in G$ the power series

$$F_V(g, z) = \sum_{n \geq 0} \chi_{S^n(V)}(g) z^n$$

- (i) Show that $F_V(g, z) = \det(1 - zg)^{-1}$ where the right hand side is the determinant of the endomorphism $1 - zg : V[z] \rightarrow V[z]$ of polynomials in V with coefficients in V .
- (ii) For each simple G -module W consider the power-series $M_V(W, z) = \sum_{n \geq 0} \langle \chi_{S^n(V)}, \chi_W \rangle z^n$. and show that

$$M_V(W, z) = \frac{1}{|G|} \sum_{g \in G} \frac{\chi_W(g^{-1})}{\det(1 - zg)}.$$

- (iii) Let $G = S_3$ and $V = \mathbb{C}^2$ the standard representation. Compute the ‘Molien power-series’ $M_V(W, z)$ for each simple module W of G .

Notice, in particular, that the determinant above is a polynomial in $\mathbb{C}[z]$, so this power-series is a *rational function*.

If $H \triangleleft G$ is a normal subgroup, U is an H -module and $g \in G$, we write ${}^gU$ for the representation of H where h acts by $ghg^{-1} \in H$ and say it is *conjugate to U* .

2. Clifford's theorem. Let H be a *normal* subgroup of G and suppose that V is a simple G -module. Then V is a semi-simple H -module, all of its irreducible components are conjugate and have the same multiplicity. Prove this result as follows:

- (i) If U is a simple H -submodule of V , show that $V = \sum_{g \in G} gU$.
- (ii) Use that H is normal to show each gU is H -invariant.
- (iii) Show that gU and U are conjugate representations of H .
- (iv) Show that G acts by permutation on the set $\mathcal{S}$ of irreducible H -modules appearing in V , and that this action is transitive.
- (v) Deduce all modules have the same multiplicity.