

Representation Theory — WiSe 2022

Bonus exercises for Tutorial 2

Real, complex and quaternionic representations. In the following exercises we explore the relation between real and complex representations of a finite group.

1. If V is a complex G -representation, the space $\text{Bil}_V = (V \otimes V)^*$ of bilinear forms on V is a G -representation. Show that if V is irreducible, then Bil_V^G has dimension 1 or 0 according to whether V is isomorphic to V^* or not.
2. Show that if $\text{Sym}^2 \text{Bil}_V \neq 0$ then $\text{Sym}^2 \text{Bil}_V^G \neq 0$, and that the same is true if we replace Sym with Alt . Conclude that if V is irreducible, then $\text{Sym}^2 \text{Bil}_V^G = 1$ implies that $\text{Alt}^2 \text{Bil}_V = 0$, and that the same is true if we replace Sym with Alt .
3. Suppose that V is irreducible and isomorphic with V^* . Then V is defined over $\mathbb{R}$ if and only if $\dim \text{Sym}^2 \text{Bil}_V^G = 1$ and $\text{Alt}^2 \text{Bil}_V = 0$. Moreover, if V is not defined over $\mathbb{R}$, then $\dim V$ is even.

We define the Schur indicator of a complex representation V to be the number

$$\text{sch}_V = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^2).$$

4. When V is irreducible, show that $\text{sch}_V \in \{0, 1, -1\}$, corresponding to the cases:
 - (i) V is not self-dual.
 - (ii) V is self-dual and defined over $\mathbb{R}$.
 - (iii) V is self-dual but not defined over $\mathbb{R}$.
5. When V is irreducible, show that $\text{sch}_V = -1$ if and only if it is obtained by restriction of scalars through the inclusion of division algebras $\mathbb{C} \subseteq \mathbb{H}$. We call such representations *quaternionic*.
6. Show that if V is irreducible then the division algebra $\text{End}_{\mathbb{R}}(V)$ corresponds to the type of V : real, complex or quaternionic.