

Representation Theory — WiSe 2022

Lecture 10: the projective, injective and simple modules of a finite dimensional algebra

During **Lecture 9**, we assigned to each finite dimensional basic connected $\mathbb{k}$ -algebra A a finite quiver Q_A with vertices in bijection with any complete set of orthogonal primitive idempotents, and with one arrow $e \longrightarrow f$ for each element in a basis of eRf where $R = \text{rad}(A)/\text{rad}(A)^2$. We proved that Q_A does not depend on the choice of such complete set of orthogonal primitive idempotents, and that eRf is isomorphic to the quotient of $e\text{rad}(A)f$ by $e\text{rad}(A)^2f$. We now aim to show that the fact A is connected implies that Q_A is connected. We begin with a technical lemma:

Lemma 1. *For each arrow $\alpha : e \longrightarrow f$ in Q_A and pick an element $x_\alpha \in e\text{rad}(A)f$ such that the classes of x_α give a basis of eRf . Then:*

(i) *For each pair of vertices e, f in Q_A and each element $x \in e\text{rad}(A)f$, we have an expression*

$$x = \sum \lambda_{\vec{\alpha}} x_{\alpha_1} \cdots x_{\alpha_\ell}$$

where the sum is taken over all paths from e to f whose arrows are one of the α s.

(ii) *Each element x_α uniquely determines a non-zero non-isomorphism $X_\alpha \in \text{hom}_A(fA, eA)$ such that $X_\alpha(f) = x_\alpha$ with the property that $\text{im}(X_\alpha) \subseteq e\text{rad}(A)$ but $\text{im}(X_\alpha) \not\subseteq e\text{rad}(A)^2$.*

Proof. To prove (i), we notice that by definition of the x_α , we can always find some linear combination $x_1 = \sum_{\alpha: e \rightarrow f} \lambda_\alpha x_\alpha$ such that $x - x_1 \in \text{rad}(A)^2$. Let us call this element y , and notice that since $\text{rad}(A)$ decomposes into the summands $e'\text{rad}(A)f'$, we can further decompose

$$e\text{rad}(A)^2f = \bigoplus_{f' \in Q_{A,0}} (e\text{rad}(A)f')(f'\text{rad}(A)f).$$

Repeating the previous argument with the elements in $(e\text{rad}(A)f')$ and $f'\text{rad}(A)f$, we deduce that there exist some linear combination $x_2 = \sum_{\alpha_1, \alpha_2: e \rightarrow f', f' \rightarrow f} \lambda_{\alpha_1 \alpha_2} x_{\alpha_1} x_{\alpha_2}$ such that $y - x_2 \in e\text{rad}(A)^3f$. Continuing in this way, and using that $\text{rad}(A)$ is nilpotent, we can write $x = x_1 + x_2 + x_3 + \cdots$. To address (ii), we notice that by hypothesis X_α corresponds to the element x_α in eAf that is in $e\text{rad}(A)f$ but not in $e\text{rad}(A)^2f$. □

Corollary 1. *If A is a basic connected finite dimensional algebra then the ordinary quiver Q_A is connected.*

Proof. Suppose that Q_A can be written as a union of two disjoint sets of vertices V and V' which are connected by no path in Q_A . It suffices we show that if $e \in V$ and $f \in V'$ then $eAf = fAe = 0$ which we know implies that A is disconnected. We notice that since fA and eA are non-isomorphic every map $\varphi : fA \rightarrow eA$ has its image in $\text{rad}(eA)$. Indeed, by I.4.5 the submodule $\text{rad}(eA) = e\text{rad}(A)$ is the unique maximal proper submodule of eA so if $\text{im}(\varphi)$ is not contained in $\text{rad}(eA)$, it follows that φ is surjective. It is easy to see now that $\ker(\varphi)$ is a direct summand of fA ¹, so it must be the zero submodule, which means that φ is an isomorphism, a contradiction. To conclude, we notice now that we have the following sequence of isomorphisms:

$$eAf \cong \text{hom}_A(fA, eA) \cong \text{hom}_A(fA, \text{rad}(eA)) \cong \text{rad}(eA)f \cong e\text{rad}(A)f$$

and by the previous lemma this last space is zero. $\square$

It is clear that if (Q, I) is a bound connected quiver and $A = \mathbb{k}(Q, I)$ then $Q_A = A$. We now show that any basic connected finite dimensional algebra A can be presented as a quotient of the path algebra $\mathbb{k}Q_A$ by an admissible ideal I .

Theorem 1. *There is a surjective $\mathbb{k}$ -algebra map $\pi : \mathbb{k}Q_A \rightarrow A$ whose kernel is an admissible ideal. Thus, A is isomorphic to the bound quiver algebra $\mathbb{k}(Q_A, I)$ with $I = \ker \pi$.*

Proof. Keeping with the notation above, we define π so that $\pi(e) = e$ and such that $\pi(\alpha) = x_\alpha$ for each arrow α in Q_A . By the universal property of $\mathbb{k}Q_A$, this is enough to define a the $\mathbb{k}$ -algebra morphism π , and it suffices to show it is surjective. To prove this, we use that $\mathbb{k}$ is algebraically closed and the *Wedderburn–Mal'cev Theorem*, which asserts that the canonical epimorphism of algebras $q : A \rightarrow A/\text{rad}(A)$ splits: there exists a semisimple subalgebra S of A such that $A = S \oplus \text{rad}(A)$ as vector spaces and $S \cong A/\text{rad}(A)$ under q . Since S is clearly generated by the idempotents of A , it suffices to show that each element of $\text{rad}(A)$ is a polynomial in the x_α , which is precisely the content of the lemma above.

We now show that $I = \ker \pi$ is admissible. If R is the arrow ideal of $\mathbb{k}Q_A$ then $\pi(R) \subseteq \text{rad}(A)$ and since $\text{rad}(A)$ is nilpotent we see that $R^n \subseteq I$ for some $n \gg 0$. To see that $I \subseteq R^2$, pick $x \in I$ and write $x = \sum \lambda_e e + \sum \mu_\alpha \alpha + y$ where $y \in R^2$. We then see that $\sum \lambda_e \pi(e) \in \text{rad}(A)$ and since these idempotents are orthogonal and $\text{rad}(A)$ is nilpotent, that $\lambda_e = 0$ for all e . Now we deduce that $\sum \mu_\alpha x_\alpha = 0$ modulo $\text{rad}(A)^2$, but we had originally picked these elements to be linearly independent modulo $\text{rad}(A)^2 = 0$, so we see that $\mu_\alpha = 0$ for all α , and hence $I \subseteq R^2$. $\square$

It follows that any finite dimensional connected basic algebra can be presented as a bound quiver algebra of a finite connected quiver. In the next lectures, we will hence focus our attention in studying the category of representations of bound quiver algebras.

¹Notice that fA is *projective*, so φ admits a section.

The projective, injective and simple modules. A sequence of composable morphisms of A -modules $f_j : M_j \rightarrow M_{j-1}$ for $j \in \mathbb{Z}$ is called a (long) exact sequence if $\ker f_j = \operatorname{im} f_{j-1}$ for all $j \in \mathbb{Z}$. More generally, we say it is a chain complex of A -modules if $\ker f_j \supseteq \operatorname{im} f_{j-1}$ for all $j \in \mathbb{Z}$. A short exact sequence is a (long) exact sequence of the form

$$S_{f,g} : 0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0.$$

A morphism of exact sequences $S \rightarrow S'$ is a triple (α, β, γ) of ‘vertical’ morphism making the obvious diagram commutative. The study of the category complexes of A -modules is part of an area of algebra called *homological algebra*, and one of its most fundamental results is the following *Snake Lemma*:

Lemma 2. Suppose that $u = (\alpha, \beta, \gamma)$ is a morphism of sequences as in the diagram below, where the top sequence is right exact and the bottom sequence is left exact. Then, there exists a unique map $\delta : \ker(\gamma) \rightarrow \operatorname{coker}(\alpha)$, called the connecting morphism associated to u , that renders the six-term “kernel–cokernel” sequence in the diagram exact.

$$\begin{array}{ccccccc}
 & \ker \alpha & \longrightarrow & \ker \beta & \longrightarrow & \ker \gamma & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow 0 \\
 & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{coker} \alpha & \longrightarrow & \operatorname{coker} \beta & \longrightarrow & \operatorname{coker} \gamma
 \end{array}$$

Moreover, the map $\ker \alpha \rightarrow \ker \beta$ is injective if $M' \rightarrow M$ is injective, and $\operatorname{coker} \beta \rightarrow \operatorname{coker} \gamma$ is surjective if $N \rightarrow N''$ is surjective, and the assignment $u \mapsto \delta$ is functorial on the category of morphisms of exact sequences.

Proof. We will define the map $\delta : \ker \gamma \rightarrow \operatorname{coker} \alpha$ by hand, and then check that the resulting sequence is exact. To make things easier to follow, let us label the maps in the first horizontal exact sequence by f and g and the maps in the second one by h and k .

Thus, suppose that we pick an element $x'' \in M''$ such that $\gamma x'' = 0$. Since the middle row is (right) exact, there exists $x \in M$ such that $g(x) = x''$. Since the square involving β and γ commutes, we see that $\beta(x) \in N$ is in the kernel of the morphism $N \rightarrow N''$, and so there is some $y' \in N'$ such that $h(y') = \beta(x)$. We define $\delta(x'') = \llbracket y' \rrbracket$, the class of y' in $\operatorname{coker} \alpha$.

We must check that $\delta(x'')$ does not depend on the choices made along the way, and that it is linear. For the first part, notice that if two $x_1, x_2 \in M$ have the same image in M'' then $x_1 - x_2$ is in the

image of f . It follows that if $f(x') = x_1 - x_2$ then $\beta(x_1) - \beta(x_2) = h\alpha(x')$. Hence, if $h(y'_1) = \beta(x_1)$ and $h(y'_2) = \beta(x_2)$, it follows that $y'_1 - y'_2 = \alpha(x')$ since h is injective and so $\llbracket y'_1 \rrbracket = \llbracket y'_2 \rrbracket$. This shows that δ is well-defined, and a similar argument shows that it is linear.

Finally, we check that the image of $\ker \beta \rightarrow \ker \gamma$ is equal to the kernel of δ , and that the kernel of $\text{coker } \alpha \rightarrow \text{coker } \beta$ is equal to the image of δ . Notice first that the definition of δ shows at once that if $x'' \in \ker \gamma$ is already covered by some $x \in M$ such that $\beta(x) = 0$ then we can choose $y' = 0$ above and so $\delta(x'') = 0$. Similarly, if we apply $N' \rightarrow N$ to $\llbracket y' \rrbracket$ we get $\llbracket \beta(x) \rrbracket = 0$, so at least we have the inclusions

$$\text{im}(\ker \beta \rightarrow \ker \gamma) \subseteq \ker \delta, \quad \text{im } \delta \subseteq \ker(\text{coker } \alpha \rightarrow \text{coker } \beta).$$

Suppose now that $\delta(x'') = 0$, and using our notation above, let $x \in M$ be such that $gx = x''$ and let $y' \in N'$ be such that $h(y') = \beta(x) \in N$. We see that $y' = \alpha(x')$ for some $x' \in M$, and so $h\alpha(x') = \beta f(x') = \beta(x)$. It follows that $x - f(x') \in \ker \beta$ and moreover that $g(x - f(x')) = gx = x''$, so x'' is in the image of $\ker \beta \rightarrow \ker \gamma$, like we wanted to show. Similarly, suppose that $\llbracket y' \rrbracket$ is such that $\llbracket hy' \rrbracket = 0$ or, what is the same, that $hy' = \beta(x)$ for some $x \in M$. If we put $x'' = g(x)$ then $\gamma(x'') = \gamma g(x) = k\beta(x) = kh y' = 0$ so $x'' \in \ker \gamma$ and it is clear, by definition of δ , that $\llbracket y' \rrbracket = \delta(x'')$. One can check directly that the six-term sequence is exact at the remaining places, and that if f is one-to-one then so is $\ker \alpha \rightarrow \ker \beta$, and similarly that if k is onto so is the induced map on cokernels. We leave the proof of functoriality to the reader, noting that it also depends on carefully following the definition of δ where we now must consider a three dimensional diagram involving 12 modules and 14 morphisms. $\square$

If M is an A -module, there is a functor $h_M(-) = \text{hom}_A(M, -)$ that assigns to each A -module N the set $\text{hom}_A(M, N)$ of A -linear maps $u : M \rightarrow N$ and to each morphism $s : N \rightarrow N'$ the map $s_* : \text{hom}_A(M, N) \rightarrow \text{hom}_A(M, N')$ such that $s_*(u) = s \circ u$. Dually, there is a functor $h^N(-) = \text{hom}_A(-, N)$ for which $t : M \rightarrow M'$ is assigned the map $t^* : \text{hom}_A(M', N) \rightarrow \text{hom}_A(M, N)$ such that $t^*(v) = v \circ t$ for any map $t : M' \rightarrow N$.

Definition 1. We say that an A -module P is projective if whenever $f : M \rightarrow M'$ is a surjective map of A -modules then the map f_* is surjective. In other words, whenever we have a diagram of the form

$$\begin{array}{ccc} & P & \\ \swarrow \text{dashed} & \downarrow h & \\ M & \xrightarrow{f} & M' \longrightarrow 0 \end{array}$$

where the horizontal map is surjective, one can complete the dashed arrow with a map $g : P \rightarrow M$ such that $fg = h$. Dually, an A -module J is injective if whenever $t : N \rightarrow N'$ is an injective map of A -modules, the map t^* is surjective. Finally, we say that an A -module is free if it is of the form $A^{(I)}$ for some indexing set I .

The following lemma establishes a fundamental relation between free and projective modules, and gives us some information about what projective modules look like once we obtain a complete set of primitive idempotents for an algebra.

Lemma 3.

- (i) *An A -module P is projective if and only if it is a direct summand of a free module, and a direct summand of a projective module is projective.*
- (ii) *If $A = e_1A \oplus \cdots \oplus e_nA$ is a decomposition of A into indecomposable modules then each e_iA is projective and every indecomposable projective A -module is isomorphic to e_iA for some $i \in [n]$.*

Notice that an algebra A is semisimple if and only if every A -module is projective. Indeed, if N is a semisimple A -module then the fact every submodule is complemented easily implies that any surjective map $M \rightarrow N$ admits a section. Picking M to be a free module as in the proof below shows that N is a direct summand of a free module. Conversely, if every A -module is projective then considering the surjection $M \rightarrow M/N$ for a submodule $N \subseteq M$ shows that N is complemented in M .

Proof. If M is any A -module and we pick the free A -module F with basis e_m for $m \in M$ and the map $\pi : F \rightarrow M$ such that $\pi(e_m) = m$, we see that every A -module admits a surjective map from a free A -module. If M is projective, then there exists a section $s : M \rightarrow F$ for π , and then $F = \ker \pi \oplus s(M)$ where $M \cong s(M)$. Thus, M is (isomorphic to) a summand of the free A -module F . It is easy to see that free modules are projective, since if F is free with basis X , one has a natural isomorphism of functors $\text{hom}_A(F, M) \cong M^{(X)}$ and if $f : M \rightarrow N$ is surjective, then clearly so is $f^{(X)} : M^{(X)} \rightarrow N^{(X)}$. Thus, suppose that P is an A -module so that there exists another A -module Q for which $F = P \oplus Q$ is free, and consider the projection $p : F \rightarrow P$. Given a diagram as in the previous definition, there exists at least a map $g' : F \rightarrow M$ such that $fg' = kp$. If we now multiply this equation by the inclusion $i : P \rightarrow F$ into the first coordinate and use that $pi = 1_P$, we see that the map $g = g'i$ has the property that $fg = h$, which is what we wanted.

To prove the second point, suppose that P is a projective module and that we decompose it into a sum of indecomposable A -modules. Following the proof above in case P is free, we see that each of these indecomposable summands is also projective. Moreover, if $P \oplus Q$ is free, it must be a sum of copies of A and thus a sum of the indecomposable A -modules e_iA for $i \in [n]$ with certain multiplicities. The uniqueness of decompositions into indecomposable summands implies then that these are precisely the isomorphism classes of indecomposable summands appearing in P , which finishes our proof. $\square$

The *standard duality* of the finite dimensional $\mathbb{k}$ -algebra A is the functor D from left A -modules to right A -modules that assigns M to the $\mathbb{k}$ -linear dual $DM = \text{hom}_{\mathbb{k}}(M, \mathbb{k})$. This functor is contravariant

and sends finitely generated A -modules to finitely generated A -modules. Moreover, it has the property that D^2 is naturally isomorphic to the identity of ${}_A\text{mod}$. Let us write ${}_A\text{proj}$ and ind_A for the full subcategories of finitely generated projective left A -modules and the full subcategories of finitely generated injective right A -modules. Then:

- (i) A sequence S is exact in ${}_A\text{mod}$ if and only if $D(S)$ is exact in mod_A .
- (ii) The functor D sends direct sum decompositions into direct sum decompositions.
- (iii) We have that $D({}_A\text{proj}) = \text{ind}_A$ and $D({}_A\text{ind}) = \text{proj}_A$.
- (iv) D induces a bijection between the simple left A -modules and the simple right A -modules.

Definition 2. A map $h : M \longrightarrow N$ is called a *minimal epimorphism* if it is onto and $\ker h$ is superfluous in M , in the sense that for any submodule $M' \subseteq M$ if $M' + \ker h = M$ then in fact $M = M'$. We say that the minimal epimorphism h is a *projective cover* if M is projective.

One can check that $h : P \longrightarrow N$ is a projective cover if and only if P is projective, and for any map $g : M \longrightarrow P$, it follows that g is onto if hg is onto. Using the lifting property of projective modules, it is not hard to see that a projective cover is unique up to isomorphism: if $p : P \longrightarrow M$ and $q : Q \longrightarrow M$ are projective covers, there exists an isomorphism $f : P \longrightarrow Q$ such that $qf = p$. Recall that the top of an A -module M is the quotient $\text{top}(M) = M/\text{rad}(M)$. Since $\text{rad}(M)$ is superfluous in M , it follows that $P \longrightarrow \text{top}(P)$ is a projective cover whenever P is projective.

Theorem 2. Suppose that $A = e_1A \oplus \cdots \oplus e_nA$ is a decomposition of A into indecomposable submodules. Then:

- (i) Any simple left A -module is isomorphic to one of the modules in the set $\{\text{top}(Ae_i) : i \in [n]\}$.
- (ii) Any indecomposable projective left A -module is isomorphic to one of the modules in the set $\{Ae_i : i \in [n]\}$ and $Ae_i \cong Ae_j$ if and only if $\text{top}(Ae_i) \cong \text{top}(Ae_j)$.
- (iii) Any indecomposable injective left A -module is isomorphic to one of the modules in the set $\{D(e_iA) : i \in [n]\}$.

Proof. If S is a simple A -module then it is in turn a simple B -module where $B = A/\text{rad}(A)$. Since B is semisimple, it follows that S is of the form $\bar{e}_iB$ for some $i \in [n]$. Since $\text{rad}(e_iA)$ is the unique maximal submodule of e_iA , it follows in fact that $\text{top}(e_iA) \cong \bar{e}_iB$. We already proved the first statement of (ii), so it suffices to show that if $\text{top}(e_iA) \cong \text{top}(e_jA)$ then $e_iA \cong e_jA$. But for any projective module P the map $P \longrightarrow \text{top}(P)$ is a projective cover, so P can be recovered from $\text{top}(P)$ up to isomorphism. The result in (iii) follows by applying D to what we have just proved. $\square$