

Representation Theory — WiSe 2022

Lecture 12: extensions, the radical of ${}_A\text{mod}$ and the Auslander–Reiten translations

The radical of ${}_A\text{mod}$. To study the category of A -modules, it proves useful to study its radical and its powers, which generalizes the usual notion of radical of an algebra that we have already introduced. Given two A -modules M and N , we define $\text{rad}_A(M, N)$ to be the set of morphisms $f : M \longrightarrow N$ with the property that for every map $h : N \longrightarrow M$, the map $1_X - hf$ is an invertible endomorphism of M . The following lemma tells us this behaves much like the radical of an algebra, which is a two sided ideal. We leave the proof for the exercises.

Lemma 1. *The radical of ${}_A\text{mod}$ is an ideal of ${}_A\text{mod}$: for each $M, N \in {}_A\text{mod}$, we have that $\text{rad}_A(M, N)$ is a subspace of $\text{hom}_A(M, N)$ with the property that:*

- (i) if $f \in \text{rad}_A(X, Y)$ and $g \in \text{hom}_A(Y, Z)$ then $gf \in \text{rad}_A(X, Z)$,
- (ii) if $f \in \text{rad}_A(X, Y)$ and $h \in \text{hom}_A(W, X)$ then $fh \in \text{rad}_A(W, Y)$.

Note: if $\mathcal{I}$ is an ideal of a $\mathbb{k}$ -linear category $\mathcal{C}$, we define the quotient $\mathcal{C}/\mathcal{I}$ to be the category with the same objects as $\mathcal{C}$ but morphisms given by $\mathcal{C}(X, Y)/\mathcal{I}(X, Y)$.

Definition 1. Any ideal $\mathcal{I}$ of ${}_A\text{mod}$ is finitely additive in both its coordinates.

Proof. If $X = X_1 \oplus \cdots \oplus X_s$ and $Y = Y_1 \oplus \cdots \oplus Y_t$ then the canonical isomorphism $F : \mathcal{C}(X, Y) = \bigoplus_{i,j=1,1}^{s,t} \mathcal{C}(X_i, Y_j)$ assigns a map $f : X \longrightarrow Y$ the maps $f_{j,i} : X_i \longrightarrow Y_j$ defined by $\pi_j^Y \circ f \circ \iota_i^X$ where π and ι denote the canonical coordinate projection and inclusions. In the same way, the inverse map assigns a ‘matrix of morphisms’ $(f_{j,i})$ to the sum of the maps $\iota_j^Y f_{j,i} \pi_i^X$. It follows that both F and its inverse restrict to the requisite bijection. $\square$

In particular, if $M = M_1 \oplus \cdots \oplus M_s$ and $N = N_1 \oplus \cdots \oplus N_t$ then the canonical isomorphism

$$\text{hom}_A(M, N) = \bigoplus_{i,j=1,1}^{s,t} \text{hom}_A(M_i, N_j)$$

restricts to a canonical isomorphism

$$\text{rad}_A(M, N) = \bigoplus_{i,j=1,1}^{s,t} \text{rad}_A(M_i, N_j).$$

Notice also that $\text{rad}_A(M, M) = \text{rad End}_A(M)$ is the usual radical of the algebra $\text{End}_A(M)$ of A -linear endomorphisms of M . The following result gives us a description of $\text{rad}_A(M, N)$ in case both M and N are indecomposable.

Proposition 1. *If M and N are indecomposable A -modules, then $\text{rad}_A(M, N)$ consists of all non-isomorphisms $f : M \rightarrow N$.*

Proof. Since M and N are finite dimensional, they are indecomposable if and only if their algebras of endomorphisms are local. It is clear that elements in the radical are not an isomorphism, so let us assume that $f : M \rightarrow N$ is a non-zero non-isomorphism. If $g : N \rightarrow M$ then $gf \in \text{End}_A(M)$ is not-invertible, for if not then we get $sgf = 1$ and then $e = fsg : N \rightarrow N$ satisfies $e^2 = e$ so it must be either zero or the identity, since $\text{End}_A(N)$ is local. But $ef = f$ so e cannot be zero, so it is e , which is a contradiction since f is not invertible. Since gf has no inverse and $\text{End}_A(M)$ is local with unique maximal ideal $\text{rad}_A(M)$ we see that $gf \in \text{rad}_A(M)$ and thus $1 - gf$ is invertible for any g , like we wanted. $\square$

The previous result allows us to give another characterization of the radical, which highlights the role of indecomposable modules when studying it.

Proposition 2. *The radical $\text{rad}_A(M, N)$ is equal to the subspace of morphisms $f : M \rightarrow N$ with the property that if $u : Z \rightarrow M$ and $v : N \rightarrow Z$ are morphisms and Z is indecomposable, then $vfu : Z \rightarrow Z$ is not an isomorphism.*

Proof. Let us write rad'_A for the subfunctor of hom_A defined by the condition of the proposition. Since $\text{End}_A(Z)$ is local whenever Z is finite dimensional, it is straightforward to check that rad'_A is an ideal of ${}_A\text{mod}$. Keeping the notation in the statement of the proposition, we know that if $f \in \text{rad}_A(M, N)$ then $vfu \in \text{rad}_A(Z, Z)$ is not an isomorphism. Now suppose that $f \in \text{rad}'_A(M, N)$ and that M is indecomposable. If $g : N \rightarrow M$ is any map then by our definition gf is not invertible in the local algebra $\text{End}_A(M)$, so $1 - gf$ is an isomorphism and $f \in \text{rad}_A(M, N)$. This is enough, since we can now use that both rad_A and rad'_A are finitely additive and that any module in ${}_A\text{mod}$ is a finite direct sum of indecomposables. $\square$

We can naturally define *powers* of the radical rad_A : for each $n \geq 2$ and each pair M, N of A -modules, we define $\text{rad}_A^n(M, N)$ to be the subspace of $\text{rad}_A(M, N)$ consisting of all finite sums of morphisms $M \rightarrow N$ of the form $f_1 f_2 \cdots f_n$ where $f_i : X_i \rightarrow X_{i+1}$ is in $\text{rad}_A(X_i, X_{i+1})$ and $X_1 = M$ and $X_{n+1} = N$. We will see later that the quotient $\text{rad}_A / \text{rad}_A^2$ will allow us to describe the category of representations of A in certain favourable cases.

Extensions. Let A be a finite dimensional $\mathbb{k}$ -algebra. A short exact sequence of left A -modules

$$\xi : 0 \rightarrow M \xrightarrow{i} E \xrightarrow{p} N \rightarrow 0$$

is called an extension of N by M . This means that M is a submodule of E (usually also referred to as the extension module) such that the quotient module E/M is isomorphic with N . Indeed i is an injection which identifies M with $i(M) \subseteq E$, and the condition that $\ker p = \operatorname{im} i$, along with the fact that p is onto gives us an isomorphism $E/i(M) \cong N$. We say two extensions ξ and ξ' of N by M are *isomorphic* if there is an isomorphism $\varphi : E \rightarrow E'$ such that $\varphi i = i'$ and $p = p' \varphi$. There always exists the *trivial* extension

$$\xi_0 : 0 \longrightarrow M \xrightarrow{i_1} M \oplus N \xrightarrow{\pi_2} N \longrightarrow 0$$

and, more generally, we say that an extension is trivial (or split) if it is isomorphic to ξ_0 . The following *Splitting Lemma* gives necessary and sufficient conditions for an extension to be split. Recall that if $f : A \rightarrow B$ and $g : B \rightarrow A$ are maps such that $gf = 1_A$, we then say that f is a section of g and that g is a retraction of f .

Lemma 2. *An extension ξ is split if and only if p admits a section, if and only if i admits a retraction.*

Given two A -modules M and N , it is not clear a priori how one can classify all isomorphism classes of extensions $\mathcal{E}(N, M)$ of N by M . It turns out that one can do this by looking at the projective modules in $A\text{-mod}$. To prove this, suppose that the sequence

$$P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

is exact and that P_0, P_1 and P_2 are projective A -modules. In this way we obtain two maps

$$\operatorname{hom}_A(P_0, M) \xrightarrow{d_0^*} \operatorname{hom}_A(P_1, M) \xrightarrow{d_1^*} \operatorname{hom}_A(P_2, M).$$

We define the Abelian group $\operatorname{Ext}_A^1(N, M)$ to be quotient $\ker d_1^* / \ker d_0^*$. Although it is not evident that this definition is independent of the sequence of projective modules and morphisms that we chose, this is the case.

Proposition 3. *There is a bijection between $\mathcal{E}(N, M)$ and the set $\operatorname{Ext}_A^1(N, M)$. In particular, if A is a finite dimensional algebra and M and N are finite dimensional A -modules, there are finitely many isomorphism classes of extension of N by M .*

Proof. Suppose that $f : P_2 \rightarrow M$ is a morphism such that $f d_1 = 0$. We can then form an extension of N by M by taking the pushout E of $P_1 \rightarrow P_0$ along f , giving us a map $i : M \rightarrow E$ whose quotient is isomorphic to N . Because $\ker d_0 = \operatorname{im} d_1$, this map is injective, and similarly one checks that the kernel of the quotient $E \rightarrow N$ is equal to $\operatorname{im} i$. We now notice that if $f = g d_0$ where $g : P_0 \rightarrow M$ then the resulting extension splits, since the map on E induced by $M \oplus P_0 \rightarrow M$ such

that $(m, p_0) \mapsto m + g(p_0)$ gives a retraction of the inclusion $i : M \rightarrow E$, so the *Splitting Lemma* applies, and it follows that we have defined a map $G : \text{Ext}_A^1(N, M) \rightarrow \mathcal{E}(N, M)$.

In the other direction, suppose that we have an extension ξ of N by M . Because P_0 is projective, there exists a map $u : P_0 \rightarrow E$ such that $pu = \varepsilon$. With this at hand, we get the map $ud_0 : P_1 \rightarrow E$ such that $pu d_0 = 0$, which means that it induces a map $P_1 \rightarrow E/\ker p = \text{coker } i$. Again, since P_1 is projective and $M \rightarrow \text{coker } i$ is onto, we obtain a map $f : P_1 \rightarrow M$ such that $if = ud_0$. Since $ifd_1 = ud_0d_1 = 0$ and since i is injective, we see that $fd_1 = 0$.

This final choice of f is unique modulo morphisms of the form gd_0 , for if u' is another map such that $pu = pu'$ and if f' is a second map such that $if' = u'd_0$ then $if - if' = (u - u')d_0$. Moreover, $u - u'$ has image in $\text{im } i$, so $f - f' = i^{-1}(u - u')$, so this map is well defined as a map to $\text{Ext}_A^1(N, M)$. A little more work shows that isomorphic extensions give equivalent elements in $\ker d_1^*$, so we get a map $F : \mathcal{E}(N, M) \rightarrow \text{Ext}_A^1(N, M)$, and it turns out that F and G are inverse bijections. $\square$

Since $\text{Ext}_A^1(N, M)$ is an Abelian group, it follows that $\mathcal{E}(N, M)$ is also an Abelian group. One can define a sum of isomorphism classes of extensions, called the *Baer sum*, in such a way that F and G are in fact isomorphisms of Abelian groups.

The Auslander–Reiten translations. Let us write $M' = \text{hom}_A(M, A)$ for the A -linear dual of a left A -module, which is thus a right A -module. We first note that if P is projective then the same is true for P' , since dualising preserves direct sums and free A -modules. The evaluation map $\text{ev}_M : M \rightarrow M''$ is a natural transformation which is an isomorphism on finitely generated projective A -modules, and thus induces a duality between proj_A and ${}_A\text{proj}$.

A minimal presentation of a module M is an exact sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$ where $P_0 \rightarrow M$ and $P_1 \rightarrow \ker d_0$ are projective covers. Recall that a projective cover of a module M is a surjective map $P \rightarrow M$ where P is projective and whose kernel K is *superfluous in P* : this means that whenever $Q \subseteq P$ is a submodule and $K + Q = P$, then in fact $Q = P$. We have checked, for example, that if P is an indecomposable projective module with simple top S , then $P \rightarrow S$ is a projective cover with kernel the superfluous submodule $\text{rad } P$ of P . We also note that:

Lemma 3. *Projective covers of a module are unique up to isomorphism: if $f : P \rightarrow M$ and $g : P' \rightarrow M$ are both projective covers, there exists an isomorphism $h : P \rightarrow P'$ such that $f = gh$.*

Proof. Since P is projective and $P' \rightarrow M$ is onto, there exists a map $h : P' \rightarrow P$ as in the statement of the theorem. Since $gh = f$ is surjective, we see that $P' = \text{im } h + \ker g$: pick $x' \in P'$ so that $g(x') = f(x)$ for some $x \in P$. Then $x' = x' - h(x) + h(x)$ and $g(x' - h(x)) = g(x') - gh(x) = 0$. Since $\ker g$ is superfluous in P' , we see that $P' = \text{im } h$ and thus h is surjective. Similarly, we can show that there exists a surjective map from P' to P , and since P' and P are finite dimensional, we deduce that they have the same dimension and that h is an isomorphism. $\square$

We define $\text{Tr}M = \text{coker}(d_1^t)$ and call it the *transpose* of M . The following lists the fundamental properties of Tr :

Proposition 4. *Let M be an indecomposable A -module.*

- (i) *The kernel of Tr consists precisely of the projective modules.*
- (ii) *If M is not projective then $P_0^t \longrightarrow P_1^t \longrightarrow \text{Tr}M$ is a minimal projective presentation of $\text{Tr}M$.*
- (iii) *Tr preserves and reflects isomorphisms between indecomposable non-projective modules.*
- (iv) *Its transpose has no non-zero projective summands.*
- (v) *If M is not projective, then $\text{Tr}M$ is indecomposable (and not projective).*
- (vi) *Tr induces a duality on indecomposable non-projectives.*

Proof. If M is projective then clearly $0 \longrightarrow M \longrightarrow M \longrightarrow 0$ is a projective presentation of M , and so $\text{Tr}M = 0$. Conversely, if $\text{Tr}M = 0$ then $P_0^t \longrightarrow P_1^t$ is surjective, which means that d_1^t is a retraction and thus that d_1 is a section (and is, in particular, injective). The splitting lemma then implies M is actually a direct summand of P_0 , so it is projective. This proves (i). Thus, in what follows, let us assume that M is not projective.

To prove (ii), pick a minimal projective presentation $Q_1 \xrightarrow{p_1} Q_0 \xrightarrow{p_0} \text{Tr}M$. Since $P_1^t \longrightarrow \text{Tr}M$ is surjective and $p_0 : Q_0 \longrightarrow \text{Tr}M$ is minimal, we can write $P_1^t = Q_0 \oplus K_1$ for some projective module K_1 in such a way that $\pi : P_1^t \longrightarrow \text{Tr}M$ restricts to $Q_0 \longrightarrow \text{Tr}M$ on Q_0 and $\pi(K_1) = 0$. Thus, $\ker \pi = \ker p_0 \oplus K_1$ has projective cover $Q_1 \oplus K_1$. It follows that we can write $P_0^t = Q_1 \oplus K_1 \oplus K_0$ for some K_0 in such a way that $P_0^t \longrightarrow P_1^t$ sends Q_1 to Q_0 via p_1 , K_1 to K_1 via the identity and K_0 to zero, i.e. the matrix of d_1^t looks as follows:

$$\begin{pmatrix} p_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Applying $(-)^t$ must get us back to the original minimal projective resolution, which now looks like

$$Q_0^t \oplus K_1^t \longrightarrow Q_1^t \oplus K_1^t \oplus K_0^t \longrightarrow M \longrightarrow 0$$

and transposes the matrix, replacing p_1 with p_1^t . It follows that the kernel of K the map onto M is precisely $\text{im}(p_1^t) \oplus K_1^t \subseteq P_0$. If $K_1^t \neq 0$, then clearly $K + N = P_0$ with $N = (Q_1^t \oplus 0 \oplus K_0^t)$ but $N \neq P_0$, which contradicts the fact that K is superfluous, so $K_1^t = 0$. To conclude, we notice that M is isomorphic to the direct sum $\text{coker}(p_1^t) \oplus K_0^t$ and, since it is indecomposable, that one of these summands must be zero. We know that $\text{coker}(p_1^t) = \text{Tr}(\text{Tr}M)$ is not zero since M is not projective, so $K_0^t = 0$, which proves that the minimal presentation of $\text{Tr}M$ is indeed what we claimed and that $\text{Tr}(\text{Tr}M) \cong M$ whenever M is non-projective indecomposable, which proves (iii) and (vi). A similar argument proves (iv): a projective summand in $\text{Tr}M$ would contradict minimality of the presentation of M . Then (v) follows: if $\text{Tr}M$ is decomposable then it decomposes into non-

projective summands say $N_1 \oplus N_2$, which would give a decomposition of M into the non-zero modules $\text{Tr}N_1$ and $\text{Tr}N_2$. $\square$

It follows that Tr does not see projectives; the following construction allows us to promote it to a duality in a different (quotient) category of ${}_A\text{mod}$.

Definition 2. The ideal $\mathcal{P}$ of ${}_A\text{mod}$ is defined so that $\mathcal{P}(M, N)$ is the set of morphisms $M \rightarrow N$ that factor through a projective module. We call the resulting quotient category ${}_A\text{mod}$, the *projectively stable* category of left A -modules.

In a similar way, we can consider $\overline{{}_A\text{mod}}$, where we kill the injective modules. Although Tr is not a functor on ${}_A\text{mod}$, it is one on ${}_A\text{mod}$, and we have the following result:

Lemma 4. *The transpose Tr defines a duality $\text{Tr} : {}_A\text{mod} \rightarrow \text{mod}_A$.*

Proof. Let us consider the category ${}_A\mathcal{C}$ whose objects are morphisms $f : P_1 \rightarrow P_0$ between finitely generated projective left A -modules and whose morphisms are those pairs of maps $u_0 : P_0 \rightarrow P'_0$ and $u_1 : P_1 \rightarrow P'_1$ making the the resulting square commutative. There is a functor ${}_A\mathcal{C} \rightarrow {}_A\text{mod}$ that assigns f to $\text{coker } f$, and a morphism (u_0, u_1) to the induced map on cokernels. By considering the projection onto ${}_A\text{mod}$, we obtain a functor $F : {}_A\mathcal{C} \rightarrow {}_A\text{mod}$, whose kernel is an ideal ${}_A\mathcal{I}$ of ${}_A\mathcal{C}$. One can prove that, given two objects f and f' , it is given by those morphisms (u_0, u_1) for which there exists a map $w : P_0 \rightarrow P'_1$ such that $f'wf = u_0f$.

Thus, we have an induced functor $F : {}_A\mathcal{C}/{}_A\mathcal{I} \rightarrow {}_A\text{mod}$ which is fully faithful and essentially surjective, and hence an equivalence of categories. Moreover, it is clear that $(-)^t$ induces a duality ${}_A\mathcal{C} \rightarrow \mathcal{C}_A$ that sends ${}_A\mathcal{I}$ to $\mathcal{I}_A$, and we can then define Tr in the same way as we did (where now we do not need to worry about taking a minimal projective resolution!) with the upshot that we get a honest functor $\text{Tr} : {}_A\text{mod} \rightarrow \text{mod}_A$ which is a duality. $\square$

Definition 3. The *Auslander–Reiten translates* of a module M are $D\text{Tr}M$ and $\text{Tr}DM$ and we write them τM and $\tau^{-1}M$, respectively.

From what we have proved for the standard duality D , we deduce that τ and τ^{-1} induce a duality between ${}_A\text{mod}$ and $\overline{{}_A\text{mod}}$, which we will later use to study ${}_A\text{mod}$ in detail.