

Representation Theory — WiSe 2022

Lecture 13: irreducible morphisms and almost-split sequences

Special morphisms. Let L , M , and N be modules in mod_A . We say that $f : L \rightarrow M$ is *left minimal* if every $h \in \text{End}_A(M)$ such that $hf = f$ is an automorphism of M . Dually, $g : M \rightarrow N$ is *right minimal* if every $k \in \text{End}_A(M)$ such that $gk = g$ is an automorphism.

Definition 1. A morphism $f : L \rightarrow M$ is called *left almost split* if f is not a section, and every morphisms out of L which is not a section factors through f : whenever $u : L \rightarrow U$ is not a section there exists a map $u' : M \rightarrow U$ such that $u'f = u$.

We define the notion of *right almost split* dually: it is a morphism that is not a retraction, and such that every morphism with the same target that is not a retraction factors through it. We say a morphism is *left minimal almost split* if it is both left minimal and left almost split, and we define a *right minimal almost split* morphism dually. These two classes of morphisms have the special property that its target (resp. source) is uniquely determined by its source (resp. target):

Lemma 1. *The target of a left minimal almost split morphism with a chosen source is uniquely determined. Dually, the source of a right minimal almost split morphism with a chosen target is uniquely determined.*

Proof. We show that if $f : L \rightarrow M$ and $f' : L \rightarrow M'$ are both left minimal almost split, there exists an isomorphism $h : M \rightarrow M'$ such that $hf = f'$. Since both f and f' are left almost split, they are not sections, and we can find $h : M \rightarrow M'$ and $h' : M' \rightarrow M$ such that $hf = f'$ and $h'f' = f$. Then $h'hf = f$ implies that $h'h$ is an automorphism of M , so both h' and h are isomorphisms. $\square$

Lemma 2. *The source of a left almost split morphism is indecomposable. Dually, the target of a right almost split morphism is indecomposable.*

Proof. Suppose that $f : L \rightarrow M$ is left almost split and that $L = L_1 \oplus L_2$ with both summands non-zero. The projections $\pi_i : L \rightarrow L_i$ for $i \in \{1, 2\}$ are not sections so there exist $u_i : M \rightarrow L_i$ such that $u_i f = \pi_i$. Then $(u_1, u_2)^t : M \rightarrow L$ is a retraction for f , which is a contradiction since f is not a section. $\square$

Notice that we can safely omit the proofs where “left” is replaced by “right”, as the standard duality automatically translates the proofs we just gave to the case of right (minimal) almost split morphisms.

Irreducible morphisms. We say that a morphism $f : M \longrightarrow N$ is *irreducible* if:

- (i) it is neither a section nor a retraction and,
- (ii) if f factorizes as $p \circ i$, then either p is a retraction or i is a section.

Notice that this notion is self-dual. Moreover, any irreducible morphism is either a monomorphism or an epimorphism, which follows by considering its factorization through its image.

Lemma 3. *If M and N are indecomposable, then $f : M \longrightarrow N$ is irreducible if and only if it belongs to $\text{rad}_A(M, N)$ but not to $\text{rad}_A^2(M, N)$.*

Proof. Suppose that $f : M \longrightarrow N$ is irreducible, and let us show that $f \in \text{rad}_A(M, N)$ and $f \notin \text{rad}_A^2(M, N)$. Since f is not an isomorphism, the first condition holds. If $f \in \text{rad}_A^2(M, N)$, then we can write $f = gh$ where $h : M \longrightarrow Z$ and $g : Z \longrightarrow N$ belong to the corresponding radicals. Since f is irreducible, either g is a retraction or h is a section. If $h' : Z \longrightarrow M$ is a retraction for h then $h'h = 1_M \in \text{rad}_A(M, M)$, which is a contradiction. Dually, if g' is a section for g then $gg' = 1_N \in \text{rad}_A(N, N)$, which cannot be. Thus, h is not a section and g is not a retraction, which contradicts the fact that f is irreducible.

Conversely, let us suppose that f is in the radical but not in its square. Since f is not an isomorphism and M and N are indecomposable, f is neither a section nor a retraction. Suppose now that $f = gh$ as before, and, decompose Z into indecomposable and write $g = (g_1, \dots, g_t)$ to get $f = \sum_{i=1}^t g_i h_i$. Because f is not in the radical square, we see that there must exist some invertible g_i or some invertible h_j in the sum. If some g_i is invertible, then g is a retraction, and if some h_j is invertible then h is a section, so f is irreducible. $\square$

Proposition 1. *Suppose that $S : 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is non-split exact. Then:*

- (i) *The monomorphism f is irreducible if and only if for every map $V \longrightarrow N$, there is either a map $M \longrightarrow V$ or a map $V \longrightarrow M$ that factors (through) g .*
- (ii) *The epimorphism g is irreducible if and only if for every map $L \longrightarrow U$, there is either a map $M \longrightarrow U$ or a map $U \longrightarrow M$ that factors (through) f .*

For the proof, let us recall that if $a : Z \longrightarrow X \longleftarrow Y : b$ are morphisms of modules, their pullback is the submodule of the product $Z \times Y$ defined by

$$Z \times_X Y = \{(z, y) \in Z \times Y : a(z) = b(y)\},$$

along with the coordinate projections $\pi_1 : Z \times_X Y \longrightarrow Z$ and $\pi_2 : Z \times_X Y \longrightarrow Y$ (which are not necessarily surjective) such that

- (i) We have that $b\pi_1 = a\pi_2$ as maps $Z \times_X Y \longrightarrow X$.

- (ii) If W is another module with maps $p_1 : W \longrightarrow Z$ and $p_2 : W \longrightarrow Y$ such that $bp_1 = ap_2$, there exists a unique map $w : W \longrightarrow Z \times_X Y$ such that $\pi_1 w = p_1$ and $\pi_2 w = p_2$.

Moreover, we have the following lemma that we will use in the proof of the proposition above.

Lemma 4. *Suppose that S is a short exact sequence as above and that $v : V \longrightarrow N$ is any module morphism. Then, there exists a diagram with exact rows with $E = M \times_N V$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & E & \xrightarrow{\pi_2} & V & \longrightarrow & 0 \\ & & \parallel & & \downarrow \pi_1 & & \downarrow v & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

Conversely, if there exists such a diagram, then $E \cong M \times_N V$ and the maps π_1 and π_2 satisfy the universal property above.

Proof. In one direction, we can always form the partially complete diagram

$$\begin{array}{ccccccc} & & M \times_N V & \xrightarrow{\pi_2} & V & & \\ & & \downarrow \pi_1 & & \downarrow v & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \longrightarrow 0 \end{array}$$

Since g is surjective, then π_2 is surjective. Indeed, given $x \in V$, there exists $m \in M$ such that $g(m) = v(x)$ since g is surjective, so that $(m, x) \in M \times_N V$ and $\pi_2(m, x) = x$. Moreover, the kernel of π_2 consists of those $(m, 0) \in M \times_N V$ with $g(m) = v(0) = 0$, which means that $\ker \pi_2 = \ker g \times 0$. It follows that the map $i : L \longrightarrow M \times_N V$ such that $i(l) = (f(l), 0)$ has image $\ker \pi_2$ and is injective since f is injective, which completes our diagram to the diagram of the statement of the lemma.

Conversely, if such diagram exists, let us call the extension appearing in the first row ξ_1 . We can repeat the construction above and get a second extension ξ_2 of V by L , and a unique map $E \longrightarrow M \times_N V$ that fits in a morphism of extensions $\xi_1 \longrightarrow \xi_2$ that is, where the other two maps are the identity of L and of V , respectively. This map is then an isomorphism, which is what we wanted. $\square$

Proof of Proposition 1. By duality, it suffices to prove only the first statement. Suppose first that f is irreducible, and consider a map $v : V \longrightarrow N$ as in the statement of the proposition. Following the proof of Lemma 4, we obtain a factorization $f = \pi_1 i$, where $i : L \longrightarrow M \times_N V$ is induced by f and π_1 is the projection to M . Since f is irreducible, either π_1 is a retraction, which gives us the requisite map $M \longrightarrow V$ after composing with π_2 , or i is a section. In this case, π_2 is a retraction by the *Splitting Lemma*, and this gives us the requisite map $V \longrightarrow M$.

Conversely, suppose that such a non-split short exact sequence exists. Then f is not a section, and it is also not a retraction (since the sequence is not split, f cannot be an isomorphism). If we have

a factorization $f = pi$ through some module E , then we can form the diagram with exact rows where $V = \text{coker } i$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & E & \xrightarrow{q} & V & \longrightarrow & 0 \\ & & \parallel & & \downarrow p & & \downarrow v & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

The map v sends the class $\llbracket x \rrbracket$ of an element in $\text{coker } q$ to the element $gp(x)$. This is well defined, for if $x - x' = i(y)$ with $y \in L$, then $pg(x) - pg(x') = gpi(y) = gf(y) = 0$. By the Lemma 4, we can very well assume that E was constructed as the pull-back of M and V along N . If there exists a map $v_1 : V \rightarrow M$ such that $gv_1 = v$ then we see that $p - v_1\pi_2 \in \ker g$, so for each $e \in E$ there exists $y \in L$ such that $(p - v_1\pi_2)(e) = f(y)$. Since f is injective, y is unique and this defines a module map $E \rightarrow L$ which is a retraction for i . If, on the other hand, there exists $v_2 : M \rightarrow V$ such that $g = vv_2$ then the map $m \mapsto (m, v_2(m))$ defines a section for $p = \pi_1$, so p is a retraction. $\square$

Corollary 1. *Irreducible monomorphisms have indecomposable cokernels. Dually, irreducible epimorphisms have indecomposable kernels.*

Proof. Suppose that $f : L \rightarrow M$ is an irreducible monomorphism and let $g : M \rightarrow N$ be its cokernel. Assume that $N = N_1 \oplus N_2$ with N_i non-zero, and write $\iota_j : N_i \rightarrow N$ the corresponding inclusions. If there exists a morphism $u_j : M \rightarrow N_i$ such that $g = \iota_j u_j$, then ι_j is also an epimorphism (since g is one) and hence an isomorphism, which contradicts the fact both summands are non-zero. Thus, the proposition implies that there must exist $v_j : N_i \rightarrow M$ such that $gv_j = \iota_j$. But then $gv = 1_N$ for $v = (v_1, v_2)$, which means that g is a retraction, and hence that f is a section, which cannot be. $\square$

Lemma 5. *If L is indecomposable, a non-zero map $f : L \rightarrow M$ is not a section if and only if $\text{im } f^* \subseteq \text{rad End}_A(L)$. Dually, if N is indecomposable then a non-zero morphism $g : M \rightarrow N$ is not a retraction if and only if $\text{im } f_* \subseteq \text{rad End}_A(N)$.*

Proof. Since L is indecomposable, its endomorphism algebra is local and its radical is its unique maximal ideal. If $\text{im } f^* \not\subseteq \text{rad End}_A(L)$ then we can find $h : M \rightarrow L$ such that hf is an isomorphism, and thus f is a section. Similarly, if f is a section then f^* hits $1_L \in \text{End}_A(L)$, which is not contained in the radical. $\square$

We are ready to prove the main theorem of the lecture, which implies that all irreducible morphisms with a given indecomposable source already appear in the unique left minimal almost split map with that given source:

Theorem 1. *Any left minimal almost split morphism is irreducible. Moreover, if $f : L \rightarrow M$ is left minimal almost split and $f' : L \rightarrow M'$ is any morphism, then f' is irreducible if and only if $M' \neq 0$ and f' is a component of f .*

Proof. Let $f : L \rightarrow M$ be left minimal almost split. It is not section by definition, and it is not a retraction since we know L is indecomposable: if it was a retraction it would be an isomorphism, and in particular a section. Suppose now that $f = pi$ factors through $i : M \rightarrow X$ and $p : X \rightarrow N$, and that i is not a section. Since i is not a section, there exists $j : M \rightarrow X$ such that $i = jf$ and thus $f = pi = pjf$. Thus, pj is an isomorphism and thus p is a retraction, so we have proved that f is irreducible.

Suppose now that $f' : L \rightarrow M'$ is irreducible, so in particular $M' \neq 0$. Since f' is not a section, there exists $h : M \rightarrow M'$ such that $f' = hf$ and, since f' is irreducible and f is not a section, h must be a retraction. If we put $M'' = \ker h$, then there exists $q : M \rightarrow M''$ such that $(h, q) : M \rightarrow M' \oplus M''$ is an isomorphism, and then $(h, q)f = (f', qf) : L \rightarrow M' \oplus M''$ is left minimal almost split.

Conversely, we must show that every direct summand of f is also irreducible. Since f is not a section, f' is not an isomorphism, and thus it is not a retraction since L is indecomposable. On the other hand, if f' is a section then f is itself a section, which is also a contradiction. Let us assume that $f' = pi$ factors through $i : M' \rightarrow X$ and $p : X \rightarrow M'$ and that i is not a section. We will show that p is a retraction, so that f' is irreducible. By construction we have that

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \\ f'' \end{pmatrix} = \begin{pmatrix} f' \\ f'' \end{pmatrix}$$

where the first matrix is a morphism $X \oplus M'' \rightarrow M' \oplus M''$. Since i is not a section, i^* has image contained in the radical of $\text{End}_A(L)$ and, similarly, since f'' is not a section, $(f'')^*$ has image contained in $\text{End}_A(L)$. The same is true then for (i, f'') , so this is not a section. Since (f', f'') is irreducible, it must be the case the first matrix is a retraction, which implies that p is a retraction, and shows that f' is irreducible. $\square$

Definition 2. We say a short exact sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is *almost split* if f is left minimal almost split and g is right minimal almost split.

It follows that in an almost split sequence, L and N are indecomposable, and the sequence is not split. In particular L is not injective and N is not projective. Moreover, the sequence is determined up to isomorphism by L , and also by N . We will prove later the very non-trivial fact that, for any indecomposable non-projective module, there is an almost split sequence ending at it, and that for any indecomposable non-injective module, there is an almost split sequence starting with it.