

# Representation Theory — WiSe 2022

## Lecture 14: the Auslander–Reiten formulas and existence of almost split sequences

---

**Proposition 1.** *For a short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

*the following are equivalent:*

- (i) *The sequence is almost split.*
- (ii) *The module  $L$  is indecomposable and  $g$  is right almost split.*
- (iii) *The module  $N$  is indecomposable and  $f$  is left almost split.*
- (iv) *The map  $f$  is left minimal almost split.*
- (v) *The map  $g$  is right minimal almost split.*
- (vi) *The modules  $L$  and  $M$  are indecomposable and the maps  $f$  and  $g$  are irreducible.*

*Proof.* The only statement in the book whose proof is incomplete is the fact that (ii) and (iii) imply (i). Thus, assume that both (ii) and (iii) hold, and let us show that  $f$  is left minimal and  $g$  is right minimal. To prove that  $f$  is left minimal, let  $h : M \rightarrow M$  be such that  $hf = f$ . There exists a unique map  $\alpha : N \rightarrow N$  that completes the following diagram to a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \longrightarrow 0 \\ & & \parallel & & \downarrow h & & \downarrow \alpha \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & M \longrightarrow 0 \end{array}$$

Indeed, if  $n \in N$  and  $n = g(m)$  then setting  $\alpha(n) = gh(m)$  works and is well defined since the top row is exact. One can check that  $\alpha$  is an isomorphism if and only if  $h$  is an isomorphism, so it suffices to show that  $\alpha$  is an isomorphism.

Since  $\alpha$  is not a retraction, because  $N$  is indecomposable, and since  $g$  is right almost split, we see that there exists  $\beta : N \rightarrow M$  such that  $g\beta = \alpha$ . Since  $gh = \alpha g$ , we see that  $h - \beta g$  has image in the kernel of  $g$ , so for each  $m \in M$  one has  $h(m) - \beta g(m) = fs(m)$  for some unique  $s(m) \in N$ . But then  $hf - \beta gf = f = fsf$ , and since  $f$  is injective we get  $sf = 1_N$ , which means  $f$  is a section, which is a contradiction. Thus,  $g$  right almost split and  $N$  indecomposable implies that  $f$  is left minimal,

and so it is left minimal almost split. Similarly, if the sequence is not split,  $L$  is indecomposable and  $f$  is left almost split, then  $g$  is right minimal. It follows that if (ii) and (iii) hold, then  $f$  is left minimal almost split and  $g$  is right minimal almost split, so the sequence is almost split.  $\square$

**The Auslander–Reiten formulas.** Let  $X$  be any module. We have a homomorphism

$$\varphi_Y^X : Y \otimes_A X^t \longrightarrow \text{hom}_A(X, Y)$$

such that  $\varphi(y \otimes f)(x) = f(x)y$ . It turns out that, to compute the morphisms in the projectively stable category  $\underline{\text{hom}}_A(X, Y)$ , we can simply compute the cokernel of  $\varphi_Y^X$ .

**Lemma 1.** *For any pair of  $A$ -modules we have the following exact sequence, natural in  $X$  and  $Y$ :*

$$Y \otimes_A X^t \xrightarrow{\varphi_Y^X} \text{hom}_A(X, Y) \longrightarrow \underline{\text{hom}}_A(X, Y) \longrightarrow 0.$$

*Proof.* Let us first pick an epimorphism  $f : P \longrightarrow Y$ . We claim that the image of the map  $f_* : \text{hom}_A(X, P) \longrightarrow \text{hom}_A(X, Y)$  is precisely the subspace  $P(X, Y)$  of morphisms  $X \longrightarrow Y$  that factor through a projective module so that  $\text{coker } f_* = \underline{\text{hom}}_A(X, Y)$ . It is clear that the image is contained in  $P(X, Y)$ , so we only need to show the other inclusion. But if  $g : X \longrightarrow Y$  factors as  $a : X \longrightarrow Q$  and  $b : Q \longrightarrow Y$  with  $Q$  projective, we can find  $h : Q \longrightarrow P$  such that  $fh = b$ . Then  $g = ba = fha$  is in the image of  $f_*$ .

To prove the claim, we notice that  $f_* \circ \varphi_P^X = \varphi_Y^X \circ (f \otimes 1_{X^t})$  as a map  $P \otimes_A X^t \longrightarrow \text{hom}_A(X, Y)$ . Since  $f \otimes 1_{X^t} : P \otimes_A X^t \longrightarrow Y \otimes_A X^t$  is surjective and since  $\varphi_P^X$  is an isomorphism (as one can check by considering the case  $P = eA$  is projective indecomposable), we see that  $\text{im } \varphi_Y^X = \text{im } f_*$ , which gives what we wanted.  $\square$

**Lemma 2.** *For any two modules  $L$  and  $M$  there is a functorial map*

$$\omega_M^L : D\text{hom}_A(L, M) \longrightarrow \text{hom}_A(M, \nu L)$$

*which is an isomorphism when  $L$  is projective.*

*Proof.* We have a map  $D\varphi_L^M : D\text{hom}_A(L, M) \longrightarrow D(M \otimes_A L^t)$  and a natural isomorphism  $\eta : D(M \otimes_A L^t) \longrightarrow \text{hom}_A(M, DL^t) = \text{hom}_A(M, \nu L)$ , which defines  $\omega_M^L$ . Since  $D$  is a duality and  $\eta$  is a natural isomorphism, and since  $\varphi_L^M$  is an isomorphism when  $L$  is projective, the same is true for  $\omega_M^L$ .  $\square$

**Lemma 3.** *Suppose that  $N$  is any  $A$ -module and that*

$$0 \longrightarrow N \longrightarrow I^0 \xrightarrow{j_0} I^1 \xrightarrow{j_1} C \longrightarrow 0$$

is an exact sequence where  $I^0$  and  $I^1$  are injective. If  $M$  is any other  $A$ -module, there is an isomorphism

$$\frac{\ker j_*^1}{\ker j_*^0} \cong \text{Ext}_A^1(M, N).$$

*Proof.* Given  $f : M \rightarrow I^1$  such that its composition with  $I^1 \rightarrow C$  is zero, we can consider the resulting module  $E$  obtained by forming the pullback of  $j^0$  along  $f$ . One can check that because  $j^1 f = 0$ , the resulting map  $E \rightarrow M$  is surjective, and its kernel is isomorphic to  $N$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{\eta} & I^0 & \longrightarrow & I^1 \longrightarrow C \\ & & \parallel & & \uparrow & \nwarrow & \uparrow f \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M \longrightarrow 0. \end{array}$$

We obtain in this way an extension of  $\xi_f$  and we can consider its class in  $\text{Ext}_A^1(M, N)$ ; let us check it is split if  $f = j^0 f'$  for some  $M \rightarrow I^0$ . In this case, if  $(i, m) \in E$  then  $i - f'(m) \in \ker j^0$  so there exists a map  $\rho : E \rightarrow M$  such that  $i - f'(m) = \eta \rho(i, m)$ , and this is a retraction for  $N \rightarrow E$ . Indeed, this last map sends  $n \in N$  to  $(\eta(n), 0)$  so  $n = \rho(\eta(n), 0)$ . More generally, one can check that if  $f_1 - f_2 = j^0 f'$ , then  $\xi_{f_1}$  and  $\xi_{f_2}$  are isomorphic. It follows that we have a map  $\ker j_*^1 / \text{im } j_*^0 \rightarrow \text{Ext}_A^1(M, N)$ .

In other other direction, for an extension as in the bottom row of the diagram above, the fact that  $I_0$  is injective allows us to construct  $\alpha : E \rightarrow I^0$  such that  $\alpha f$  is the map  $N \rightarrow I^0$ . In turn, we can construct the following diagram of solid arrows, where the dotted arrow  $f$  is uniquely determined by lifting through  $g : E \rightarrow M$ , i.e. if  $m = g(e)$  then we set  $f(e) = j^0 \alpha(e)$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 \longrightarrow C \\ & & \parallel & & \uparrow \alpha & & \uparrow f \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \end{array}$$

By construction the map  $f$  has zero composition with  $j^1$ , and hence defines an element in the quotient  $\ker j_*^1 / \text{im } j_*^0$ . If the sequence is split, one can choose  $\alpha$  to be zero on the summand  $M$  of  $E \cong M \oplus N$  and hence  $f$  will be the zero map. More generally, one checks that if two sequences  $\xi$  and  $\xi'$  are equivalent, the resulting maps  $f$  and  $f'$  are such that  $f - f'$  factors through  $j_0$ . Finally, one checks that the resulting map  $\text{Ext}_A^1(M, N) \rightarrow \ker j_*^1 / \text{im } j_*^0$  is the inverse isomorphism to  $\llbracket f \rrbracket \mapsto \llbracket \xi_f \rrbracket$ .  $\square$

**Theorem 1** (Auslander–Reiten formulas). *For  $M$  and  $N$  two left  $A$ -modules, we have functorial isomorphisms*

$$\text{Ext}_A^1(M, N) \cong \underline{\text{Dhom}}_A(\tau^{-1}N, M) \cong \overline{\text{Dhom}}_A(N, \tau M).$$

*Proof.* Let us prove the first isomorphism holds, noticing that the second one follows from the

isomorphism  $\underline{\text{hom}}_A(X, Y) \cong \overline{\text{hom}}_A(\tau X, \tau Y)$  valid for any two  $A$ -modules  $X$  and  $Y$ , since  $\tau$  and  $\tau^{-1}$  give inverse equivalences between the projectively and injectively stable categories of  $A$ -modules. Since  $\text{Ext}_A^1(M, N) = 0$  when  $M$  is projective or when  $N$  is injective, since  $\tau^{-1}N = 0$  whenever  $N$  is injective, we can assume that  $N$  has no injective direct summand. In this case we may write  $N = \tau L$  so that  $L = \tau^{-1}N$ . Let us now take a minimal projective presentation  $P_1 \rightarrow P_0 \rightarrow L \rightarrow 0$  and consider the following commutative diagram where the vertical arrows are induced by the map  $\omega$  as above, and the horizontal arrows by the ones in the presentation:

$$\begin{array}{ccccccc}
 & & & & \ker \omega_M^L & & \\
 & & & & \downarrow & & \\
 D\text{hom}_A(P_1, M) & \longrightarrow & D\text{hom}_A(P_0, M) & \longrightarrow & D\text{hom}_A(L, M) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \text{hom}_A(M, \nu P_1) & \xrightarrow{q_1} & \text{hom}_A(M, \nu P_0) & \xrightarrow{q_0} & \text{hom}_A(M, \nu L) & & 
 \end{array}$$

Notice that the first two vertical arrows are isomorphisms since  $P_1$  and  $P_0$  are projective. Let us define a map  $\ker \omega_M^L \rightarrow \ker q_0 / \text{im } q_1$  as follows: given an element in the kernel of  $\omega_M^L$ , pick any  $x \in D\text{hom}_A(P_0, M)$  that covers it (which exists, since the top row is right exact) and then map this element down to  $\text{hom}_A(M, \nu P_0)$ . Since our original element is in  $\ker \omega_M^L$ , this element is in  $\ker q_0$ , and we take its class modulo  $\text{im } q_1$ . This map is an isomorphism: given an element in  $\ker q_0$ , map it to  $D\text{hom}_A(P_0, M)$  through the inverse of the vertical isomorphism, and then to  $D\text{hom}_A(L, M)$ . Since the rightmost square commutes, this element is in the kernel of  $\omega_M^L$ , and since the leftmost map is an isomorphism, too, the kernel of this map is  $\text{im } q_1$ . This defines an inverse to the map above, giving us the desired inverse isomorphism. We now observe that  $\ker \omega_M^L$  is isomorphic with  $D\text{coker } \phi_M^L \cong \underline{\text{hom}}_A(L, M)$  so it suffices to note that  $\ker q_0 / \text{im } q_1 \cong \text{Ext}_A^1(M, \tau L)$  since

$$0 \rightarrow \tau L \rightarrow \nu P_1 \rightarrow \nu P_0 \rightarrow \nu L \rightarrow 0$$

is the beginning of an injective resolution of  $\tau L$ , so **Lemma 3** applies.  $\square$

**Existence of almost split sequences.** We can now use the Auslander–Reiten formula to prove that almost split sequences exist. To do it, let us consider the functor  $S(L, M) = \text{hom}_A(L, M) / \text{rad}_A(L, M)$  and notice that if  $L$  is indecomposable and non-projective and  $M$  is indecomposable, then  $P(L, M) \subseteq \text{rad}_A(L, M)$ . Indeed, if  $L \rightarrow N$  is a morphism that factors through a projective, it cannot be an isomorphism, for then  $L$  would be projective. Thus, in such a situation we have a canonical surjective map  $p_{L, M} : \underline{\text{hom}}_A(L, M) \rightarrow S(L, M)$ .

When  $L = M$  we see that  $\text{hom}_A(M, M) = \underline{\text{End}}_A(M)$  is an  $\text{End}_A(M)$ -module and the surjective map  $p_{M, M}$  exhibits  $S(M, M)$  as the simple socle of  $\underline{\text{End}}_A(M)$  since  $\text{End}_A(M)$  is local. To be more precise, since  $\mathbb{k}$  is algebraically closed,  $S(M, M)$  is a one dimensional simple  $\text{End}_A(M)$ -module.

Dualizing, we thus see that the image of  $DS(M, M)$  under the injective map  $Dp_{M, M}$  is the simple socle of the module  $D\text{End}_A(M)$ . With this at hand, we can now give the proof of the main result of this lecture.

**Theorem 2.** *Suppose that  $M$  is indecomposable non-projective. Then, there exists an almost split exact sequence of the form*

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0$$

*Dually, if  $N$  is indecomposable non-injective, there exists an almost split exact sequence of the form*

$$0 \longrightarrow N \longrightarrow E \longrightarrow \tau^{-1}N \longrightarrow 0$$

*Proof.* Let  $\xi'$  be any non-zero element in  $DS(M, M)$  and let  $\xi$  be its image in  $D\text{hom}_A(M, M)$ . By the Auslander–Reiten formula, this is isomorphic with  $\text{Ext}_A^1(\tau M, M)$ , and we claim that if  $\xi$  is represented by a short exact sequence

$$0 \longrightarrow \tau M \longrightarrow E \xrightarrow{g} M \longrightarrow 0$$

then this sequence is almost split. It is at least not split since  $\xi$  is not zero and  $\tau M$  is indecomposable, so it suffices to show that  $g$  is right almost split. It is at least not a retraction since  $\xi$  is not zero, so let us pick a homomorphism  $v : V \longrightarrow M$  where  $V$  is indecomposable and let us show it factors  $g$  if it is not a retraction. In particular  $v$  is not an isomorphism so  $v \in \text{rad}_A(V, M)$  and by functoriality we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & DS(M, M) & \longrightarrow & D\text{hom}_A(M, M) & \xrightarrow{\cong} & \text{Ext}_A^1(M, \tau M) \\ & & \downarrow v_* & & \downarrow v_* & & \downarrow v^* \\ 0 & \longrightarrow & DS(M, V) & \longrightarrow & D\text{hom}_A(M, V) & \xrightarrow{\cong} & \text{Ext}_A^1(V, \tau M). \end{array}$$

Since  $v$  is in the radical its action on the semi-simple socle is zero, and thus  $v_*(\xi') = 0$ . It follows that  $v^*(\xi) = 0$  and thus the diagram obtained by taking the pull-back through  $v$  has top row that splits:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau M & \xrightarrow{f'} & E' & \xrightarrow{g'} & V \longrightarrow 0 \\ & & \parallel & & \downarrow w & & \downarrow v \\ 0 & \longrightarrow & \tau M & \xrightarrow{f} & E & \xrightarrow{g} & M \longrightarrow 0 \end{array}$$

If  $g''$  is a section for  $g'$  then  $gw g'' = v$ , and thus  $g$  is right almost split. □

*Note:* one can in fact show that if  $\xi$  is almost split, then it must belong to the simple socle of  $\text{Ext}_A^1(M, \tau M)$ . Since we know this vector space is one dimensional, it follows that there exists up

to equivalence one almost split sequence starting and  $\tau M$  and ending at  $M$ .

**Corollary 1.** *If  $M$  is indecomposable non-projective then  $\underline{\text{End}}_A(M)$  is a division algebra if and only if  $\overline{\text{End}}_A(\tau M)$  is a division algebra. In this case we have that  $\underline{\text{End}}_A(M) \cong \mathbb{k}$  and any non-split extension of  $M$  by  $\tau M$  is almost split.*

*Proof.* Since  $\tau$  gives an equivalence between the corresponding stable categories, the two stable endomorphism algebras are isomorphic. Moreover, since  $\mathbb{k}$  is algebraically closed, they must be isomorphic to  $\mathbb{k}$  as soon as they are division algebras. Finally, the Auslander–Reiten formula tells us that  $\text{Ext}_A^1(M, \tau M) \cong \mathbb{k}$ . This module is then equal to its (simple) socle and thus there is a unique non-split extension of  $\tau M$  by  $M$  which is must be almost split by the proof above.  $\square$

We say that an  $A$ -module  $M$  is a *brick* if  $\text{End}_A(M) = \mathbb{k}$ . In particular  $M$  is indecomposable. Thus, if  $M$  is a non-projective brick then  $\underline{\text{End}}_A(M) = \mathbb{k}$  and hence that any non-split extension of  $\tau M$  by  $M$  gives a representative for the unique almost split exact sequence starting at  $\tau M$  and ending at  $M$ .