

Representation Theory — WiSe 2022

Lecture 15: the Auslander–Reiten quiver of an algebra

The following five results are there to give us information about the way in which irreducible morphisms appear in the category of modules.

Proposition 1. *Suppose that X is indecomposable.*

- (i) *If X is projective, a map $g : M \rightarrow X$ is right minimal almost split if and only if it is a monomorphism with image $\text{rad}X$.*
- (ii) *Dually, if X is indecomposable injective, then $f : X \rightarrow M$ is left minimal almost split if and only if it is an epimorphism with kernel equal to $\text{soc}X$.*

Proof. Let us show that $i : \text{rad}X \rightarrow X$ is right minimal almost split whenever X is projective. Since i is a monomorphism, it is right minimal, and it is not a retraction since it is not surjective. If $v : V \rightarrow X$ is not a retraction it is not surjective (since X is projective) and thus its image must be contained in $\text{rad}X$, which we know is the maximal submodule of the indecomposable projective module X . $\square$

Corollary 1. *let X be indecomposable. There exists a right minimal almost split morphism $g : M \rightarrow X$. Moreover $M = 0$ if and only if X is simple projective. Dually, there exists a left minimal almost split morphism $X \rightarrow M$ and $M = 0$ if and only if X is simple injective.*

Proof. If X is indecomposable non-projective we know that there exists an almost split sequence ending at X , and this gives such a right minimal almost split morphism. If X is projective then we just proved the unique (up to isomorphism) right minimal almost split morphism ending at X is the inclusion of the radical, and in this case $\text{rad}X = 0$ if and only if X is simple. $\square$

Proposition 2. *If M is indecomposable non-projective, there exists an irreducible morphism of the form with target M if and only if there exists an irreducible morphism with source τM .*

Proof. Suppose that $X \rightarrow M$ is irreducible. We know that this irreducible morphism can be completed to a right minimal almost split morphism $g : X \oplus Y \rightarrow M$. Since M is not projective and there exists, up to isomorphism of the source, a unique right minimal almost split morphism ending at M , the map g must be surjective as it appears in an almost split sequence ending at M .

We then know that $\ker g = L \cong \tau M$ is indecomposable and the resulting sequence exact sequence is almost split by **Theorem 1**. Thus, the composition $\tau M \longrightarrow X \oplus Y \longrightarrow X$ is irreducible (being a factor of a left minimal almost split map), which is what we wanted. Notice that the steps are reversible, which gives us the converse: starting with an irreducible map $\tau M \longrightarrow X$, we can complete it to a left minimal almost split map $\tau M \longrightarrow X \oplus Z$, whose cokernel must be isomorphic with M , and for which the map $X \oplus Z \longrightarrow M$ is right minimal almost split, giving us an irreducible morphism $X \longrightarrow M$. $\square$

Corollary 2. *If S is simple projective and non-injective and if $f : S \longrightarrow M$ is irreducible, then M is projective. Dually, if S is simple injective and non-projective and if $g : M \longrightarrow S$ is irreducible, then M is injective.*

Proof. If M is also indecomposable, and if we assume that it is non-projective, then we know that there exists an irreducible map $f' : \tau M \longrightarrow S$ with $\tau M \neq 0$. But we also know that the image of this map must land in $\text{rad } S = 0$, which is a contradiction: f' is irreducible but neither injective nor surjective. Thus, it suffices that we show that if $M = M_1 \oplus \cdots \oplus M_t$ is the decomposition of M into indecomposables then each map $S \longrightarrow M_i$ is irreducible. Since S is not injective, there exists a left minimal almost split map $S \longrightarrow N$ which contains all irreducible morphisms out of S as a direct summand. In particular, we know that M is a direct summand of N , so that all the resulting maps $S \longrightarrow M_i$ are also irreducible. It follows that each indecomposable summand of M is projective, so M is projective. $\square$

Corollary 3. *Let S be simple projective non-injective and let P be the direct sum of all projectives P' for which S is a direct summand of $\text{rad } P'$. Then the exact sequence*

$$0 \longrightarrow S \xrightarrow{i} P \longrightarrow \text{coker } i \longrightarrow 0$$

is almost split, where i is the sum of all the inclusions $S \longrightarrow \text{rad } P'$ as a summand.

Proof. Since S is simple non-injective there exists a left minimal almost split map $S \longrightarrow P$ with P projective (since this map is, in particular, irreducible). If $P = P_1 \oplus \cdots \oplus P_t$ is the decomposition of P into indecomposable modules, then for each $i \in [t]$ the map $S \longrightarrow P_i$ is irreducible and thus is a summand of the unique right minimal almost split map $\text{rad } P_i \longrightarrow P_i$, so that each $S \longrightarrow P_i$ is the inclusion to a summand of $\text{rad } P_i$. Since i is left minimal almost split, its cokernel is indecomposable. Thus the sequence of the lemma has i left minimal almost split and $\text{coker } i$ indecomposable, and thus it is almost split by **Proposition 1**. $\square$

Proposition 3. *Let P be indecomposable projective-injective with socle S and radical R . Then the following exact sequence is almost split:*

$$0 \longrightarrow R \xrightarrow{q \oplus i} R/S \oplus P \xrightarrow{-j \times p} P/S \longrightarrow 0$$

Proof. Since P is injective it has simple socle S , and since R also has simple socle S , it is indecomposable. It follows that $R \rightarrow P$ is irreducible and in fact the unique irreducible morphism ending in P : since $R \rightarrow P$ is right minimal almost split, any irreducible morphism must appear as a summand, but R is indecomposable. Dually, P/S is indecomposable and the projection $P \rightarrow P/S$ is, up to isomorphism, the unique non-trivial irreducible morphism starting at P . By **Proposition 3**, we see that $\tau(P/S) \cong R$. The sequence is not split, since in this case P/S would be an indecomposable summand appearing in $R/S \oplus P$. Since $\dim_{\mathbb{k}} R/S < \dim_{\mathbb{k}} P/S$ it would have to be a summand of P , but P is indecomposable and $\dim_{\mathbb{k}} P/S < \dim_{\mathbb{k}} P$. Thus, by **Proposition 1**, it suffices to show that the inclusion (qi) is left almost split.

Suppose that $u : R \rightarrow U$ is not a section and decompose $U = U_1 \oplus \cdots \oplus U_t$. Then each map $u_i : R \rightarrow U_i$ is not a section: if there is $r : U_i \rightarrow R$ such that $ru_i = 1_R$ then $r\pi_i$ is a retraction for u . Thus, we can assume that U is indecomposable and show each components u_i factors through $q \oplus i$. If u is a monomorphism, then since P is injective, there exists $h : U \rightarrow P$ such that $i = hu$ where i is the inclusion to the radical of P . Since i is right minimal almost split, it is irreducible, and hence h is a retraction since we assumed that u is not a section. But U is indecomposable so h must be an isomorphism, and thus u is (isomorphic to) the inclusion into the radical, and in particular factors through $q \oplus i$. If u is not a monomorphism, then factor $u = sr$ where $r : R \rightarrow R/\ker u$ is the projection and $R/\ker u \rightarrow U$ is a monomorphism. Since $\ker u = \ker r \neq 0$, the simple socle S of R is contained in $\ker r$ and hence r factors through R/S , so that u factors through $q \oplus i$. $\square$

The Auslander–Reiten quiver. Let A be a finite dimensional algebra and let M and N be two indecomposable A -modules. We define the *space of irreducible morphisms from M to N* as the vector space quotient $\text{rad}_A(M, N)/\text{rad}_A^2(M, N)$ and write it $\text{Irr}_A(M, N)$. Notice that this is an $\text{End}_A(M)$ - $\text{End}_A(N)$ -bimodule, on which the respective radicals act by zero. The following results tells us that left (respectively, right) minimal almost split morphisms, and thus almost split sequences, give us a basis for this space.

To state it, let us consider a morphism $f : L \rightarrow M$ with L indecomposable, and let us decompose M into indecomposable modules $M_1, \dots, M_t$ with corresponding multiplicities $(a_1, \dots, a_t)$. Thus, for each index i we have a_i corresponding maps $f_{ij} : L \rightarrow M_i$ for $j \in [a_i]$. The result we are after is then:

Proposition 4. *A morphism $f : L \rightarrow M$ with L indecomposable is left minimal almost split if and only if*

- (i) *For each $i \in [t]$ and each $j \in [a_i]$ the morphism f_{ij} belongs to $\text{rad}_A(L, M_i)$.*
- (ii) *For each $i \in [t]$ the classes of the a_i morphisms $f_{i,1}, \dots, f_{i,a_i}$ form a basis of $\text{Irr}_A(L, M_i)$.*
- (iii) *If M' is indecomposable and $\text{Irr}_A(L, M') \neq 0$, then M' is isomorphic to one of the modules in the set $\{M_1, \dots, M_t\}$.*

There is of course a dual statement for right minimal almost split. Notice in particular that if now $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is almost split, we have that $L = \tau N$ and we deduce that for each indecomposable summand M_i of M ,

$$\dim_{\mathbb{k}} \text{Irr}_A(\tau N, M_i) = \dim_{\mathbb{k}} \text{Irr}_A(M_i, N),$$

giving us a stronger statement than that in **Proposition 2**.

Definition 1. The Auslander–Reiten quiver of A is the quiver whose vertices are isomorphism classes of indecomposable A -modules, for which there are as many arrows $[M] \longrightarrow [N]$ as vector in a basis of $\text{Irr}_A(M, N)$, and we write it $\Gamma(A \text{ mod})$.

Let us briefly consider the computation of the Auslander–Reiten quiver of the path algebra of the quiver $1 \longrightarrow 2 \longrightarrow 3$. One can directly prove that all indecomposable A -modules already appear within the list of 6 injectives, projectives or simples. The module $P(3)$ is projective, simple and non-injective, so that if $P(3) \longrightarrow P$ is irreducible then P is projective. Moreover, this map must be a summand of $\text{rad } P \longrightarrow P$, since $P(3) = \text{rad } P(2)$, we see that the only irreducible out of $P(2)$ is the radical inclusion $P(3) \longrightarrow P(2)$ and we have an almost split sequence which we depict as follows, where the horizontal arrow denotes an application of τ :

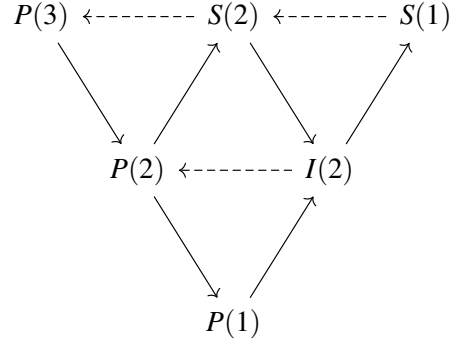
$$\begin{array}{ccc} P(3) & \xleftarrow{\quad\quad\quad} & S(2) \\ & \searrow \quad \nearrow & \\ & P(2) & \end{array}$$

To continue, we notice that $\text{rad } P(1) = P(2)$, that $P(1) = I(3)$ is injective-projective so we have an almost split sequence starting with $P(2)$ and ending with $P(1)/S(2) = I(2)$ and with middle term $P(1) \oplus S(2)$. We add this information to our quiver, which now looks as follows:

$$\begin{array}{ccccc} P(3) & \xleftarrow{\quad\quad\quad} & S(2) & & \\ & \searrow & \nearrow & \searrow & \\ & & P(2) & \xleftarrow{\quad\quad\quad} & I(2) \\ & & \searrow & \nearrow & \\ & & & P(1) & \end{array}$$

Finally, we see that the $I(2)/S(2) = S(1)$ and that $I(2) \longrightarrow S(1)$ is irreducible (since it is left minimal almost split). Moreover, since one easily computes that $\tau S(1) = S(2)$, and since there

is exactly there one irreducible morphism $I(2) \longrightarrow S(1)$, we see that $S(2) \longrightarrow I(2)$ is the unique irreducible morphism between these modules. We see that we have an almost split sequence that we can add to finish the computation of the quiver:



We make the observation that for this algebra, there are finitely many isomorphism classes of indecomposable modules, that is, this algebra has *finite representation type*, and for each pair of indecomposable modules, there is at most *one* irreducible arrow between them. The following result explains why this is the case:

Lemma 1. *If an algebra has finite representation type, then its Auslander–Reiten quiver has no double (parallel) arrows.*

Proof. Let A be a finite dimensional algebra and us assume that there exist two indecomposable modules M and N and at least two irreducible morphisms $M \longrightarrow N$ whose classes are linearly independent in $\text{Irr}_A(M, N)$. Since irreducible morphisms are either injective or surjective and since M and N are not isomorphic, either $m_0 = \dim_{\mathbb{k}} M > \dim_{\mathbb{k}} N = n_0$ or the opposite equality happens. Without loss of generality, assume the first case holds, and notice that N cannot be projective since an irreducible morphism $M \longrightarrow N$ is not a section. It follows that we have an almost split sequence of the form

$$0 \longrightarrow \tau N \longrightarrow M^2 \oplus E \longrightarrow N \longrightarrow 0.$$

Taking dimensions, we see that $n_1 = \dim_{\mathbb{k}} \tau N = 2m_0 + \dim_{\mathbb{k}} E - n_0 > m_0 > n_0$. Since there are at least two irreducible morphisms $\tau N \longrightarrow M$, we can repeat the process above, and deduce that $n_i = \dim_{\mathbb{k}} \tau^i N$ is strictly increasing. In this way, we obtain an infinite family of non-isomorphic indecomposable modules, so that A is not representation finite. $\square$