

Representation Theory — WiSe 2022

Complement to Lecture 3

Here we complement some results in **Lecture 3**. Let D be a division algebra and let V be a D -module. Let us say that a set L of D -linear transformations over V is *dense* if for any D -linearly independent set of vectors $x_1, \dots, x_n$ of V and any corresponding list of vectors $y_1, \dots, y_n$, there exists $f \in L$ such that $f(x_i) = y_i$ for all $i \in [n]$.

Theorem 1. *A $\mathbb{k}$ -algebra A admits a faithful simple module if and only if it is isomorphic to a dense ring of linear transformations of a vector space over a division $\mathbb{k}$ -algebra.*

Proof. Suppose first that A admits a faithful simple module V with structure map $\rho : A \longrightarrow \text{End}(V)$. By Schur's Lemma, $D = \text{End}_A(V)$ is a division algebra, and A is isomorphic to $\rho(A) \subseteq \text{End}_D(V)$ since the action is faithful. Pick $x_1, \dots, x_n$ a D -linearly independent set in V and any other $y_1, \dots, y_n$. We can extend $x_1, \dots, x_n$ to a basis of V so there exists a D -linear transformation T such that $Tx_i = y_i$ for each $i \in [n]$. Since T is D -linear and $D = \text{End}_A(V)$, the density theorem proved in class shows that there is some $a \in A$ for which $Tx_i = ax_i = y_i$ for all $i \in [n]$, which means that $\rho(A)$ is dense.

Conversely, if A is (isomorphic to) a dense algebra of linear transformations of a vector space V over a division algebra D then V becomes an A -module, and V is simple since the action is transitive. Thus V is a faithful simple A -module. $\square$

Notice that if V is *finite dimensional* then the only set of linear transformations that is dense is the set of all linear transformations: pick a basis $x_1, \dots, x_n$ of V , a linear transformation T and set $y_i = Tx_i$. Then there exists $f \in L$ such that $f(x_i) = y_i = Tx_i$ on a basis, so $f = T$. Thus, we have the following:

Corollary 1. *Let A be an algebra.*

- (i) *If V is a finite dimensional simple A -module and let $D = \text{End}(V)$ be the corresponding division algebra. Then the map $\rho : A \longrightarrow \text{End}_D(V)$ is onto.*
- (ii) *More generally, if V is a direct sum of pairwise non-isomorphic simple modules S , the following sum of structure maps is onto:*

$$\bigoplus_{V \in S} \rho_V : A \longrightarrow \bigoplus_{V \in S} \text{End}_{D_V}(V).$$

The following gives the structure of Artinian algebras that admit a faithful simple representation. Recall that an algebra is left Artinian if it satisfies the descending chain condition on left ideals. It is called simple if it has no non-trivial two-sided ideals.

Theorem 2. *The following are equivalent for an algebra A :*

- (i) *A is left Artinian and simple.*
- (ii) *A is left Artinian and admits a faithful simple representation.*
- (iii) *A is isomorphic to $\text{End}_D(V)$ where V is finite dimensional over a division algebra D .*

Proof. That 1. implies 2. is clear since A is faithful. Second, 2. implies 3. for if A is Artinian and V is faithful and simple then it must be finite dimensional: if not, pick some countably infinite set $X = \{x_1, x_2, \dots\}$ of D -linearly independent vectors and consider I_n the ideal of elements in A acting by zero on $x_1, \dots, x_n$. By the density theorem one has that $I_n \supsetneq I_{n+1}$ which violates the Artinian condition, since we can always find some $a \in A$ for which $ax_i = 0$ for $i \leq n$ but $ax_{n+1} \neq 0$.

If we pick a basis $x_1, \dots, x_n$ of V over D , the density theorem implies that for any $y_1, \dots, y_n \in V$ we can find $a \in A$ such that $ax_i = y_i$ for $i \in [n]$. Thus if ρ is the structure map of V we get that $\rho(A) = \text{End}_D(V)$, giving the desired isomorphism. The last condition implies the first since matrix rings are simple and left Artinian. $\square$

Theorem 3. *For a finite dimensional algebra, the following are equivalent.*

- (i) *A is semi-simple.*
- (ii) *A has zero radical.*
- (iii) *A is a product of endomorphism algebras of finite dimensional vector spaces over division algebras.*

Proof. 1. implies 2. since A is faithful. On the other hand, we know that A has finite many isomorphism classes of simples, and if S contains one simple module V for each isomorphism class, that the map

$$\bigoplus_{V \in S} \rho_V : A \longrightarrow \bigoplus_{V \in S} \text{End}_{D_V}(V)$$

is onto. Since its kernel is the Jacobson radical (as we have included all simples in the sum) we conclude the map is an isomorphism and that 2. implies 3. Finally, 3. implies 1. as one can prove by hand that products of matrix algebras are semi-simple. $\square$