

Representation Theory — WiSe 2022

Details for Lecture 4

Lemma 1. Let $C_1, \dots, C_r$ be all conjugacy classes in G , and put $z_i = \sum_{g \in C_i} g$. Then $\{z_1, \dots, z_r\}$ forms a basis of the center of $\mathbb{K}G$, and there exists natural numbers m_{ij}^k for which

$$z_i z_j = \sum_{k=1}^r m_{ij}^k z_k$$

Proof. We consider the action of G on itself by conjugation, whose orbits are the conjugacy classes and for which $(\mathbb{K}G)^G = Z(\mathbb{K}G)$ to obtain the first part. For the second part, we see that m_{ij}^k counts the number of pairs $(g_1, g_2) \in C_i \times C_j$ such that $g_1 g_2 \in C_k$, hence the result. $\square$

Lemma 2. Let V be a complex irreducible representation and C a conjugacy class, and put $z_{V,j} = |C_j| \chi_V(C_j) / \dim(V)$. Then

$$z_{V,i} z_{V,j} = \sum_{k=1}^r m_{ij}^k z_{V,k}$$

Proof. Let ρ_V be the structure map of V and pick $z \in Z(\mathbb{K}G)$. Then $\rho_V(z)$ is in $\text{End}_G(V)$ (since z is central) so there exists $\lambda \in \mathbb{C}$ for which $\rho_V(z) = \lambda \cdot 1_V$ and $\chi_V(z) = \lambda \cdot \dim(V)$ so $\lambda = \chi_V(z) / \dim(V)$. Thus, applying ρ_V to the equation of the first lemma and then taking traces, we obtain the necessary equation if we notice that $\chi_V(z_i) / \dim(V) = z_{V,i}$. $\square$

Theorem 1 (Second orthogonality relations). For $g, h \in G$, we have that

$$|C_G(g)|^{-1} \sum_{i=1}^r \chi_i(g) \overline{\chi_i(h)} = \begin{cases} 1 & \text{if } g \text{ and } h \text{ are conjugate.} \\ 0 & \text{else,} \end{cases}$$

where the sum ranges through all irreducible complex characters.

Proof. Let us consider the matrix X such that $X_{ij} = \chi_i(C_j)$ where $C_1, \dots, C_r$ are the conjugacy classes in G . At the same time, let $g_1, \dots, g_r$ be representatives for these conjugacy and consider $D = \text{diag}(d_1, \dots, d_r)$ where $d_i = |C_G(g_i)|$ is the size of the centralizer of g_i in G . The usual orthogonality relations say that $XD X^\dagger$ is the identity matrix, since for each conjugacy class C_i we have that $|C_i| = [G : C_G(g_i)]$. Thus, we also have that $X^\dagger X = D^{-1}$, which is what we wanted. $\square$