

Representation Theory — WiSe 2022

Notes for Lecture 5

Induction and restriction. Given a map of groups $\phi : G \mapsto H$, every H -module becomes a G -module, and we get a functor ${}_H\text{Mod} \rightarrow {}_G\text{Mod}$ written ϕ^* . We call it the *pullback functor* associated to ϕ . Notice that the map ϕ need not be an inclusion. For example if $N = \ker \phi$ and ϕ is onto, then a G -module V is the pullback of an H -module if and only if N acts by zero on V .

There also exists a functor ϕ_* defined on G -modules V by the formula $\phi_*(V) = \mathbb{k}H \otimes_{\mathbb{k}G} V$. Since $\mathbb{k}H$ has a right G -action and a left H -action that commute, $\phi_*(V)$ is a left $\mathbb{k}H$ -module by the formula $h'(h \otimes v) = h'h \otimes v$. More classically, for W in ${}_H\text{Mod}$ the module $\phi^*(W)$ is called the restriction of W from G to H and written $\text{Res}_G^H(W)$, while $\phi_*(V)$ is called the induced representation from G to H , and written $\text{Ind}_G^H(V)$. We will use this notation in what follows.

Suppose now H is a subgroup of G , and let us consider $\mathbb{k}G$ as a right $\mathbb{k}H$ -module. There exist coset representatives $1 = g_0, \dots, g_r$ in G such that any element in G can be uniquely written as $g_i h$ for some h in H . That is:

Lemma 1. *The right $\mathbb{k}H$ -module $\mathbb{k}G$ is free with basis $\{g_0, \dots, g_r\}$. Thus, for any H -module V , we have a decomposition into H -submodules:*

$$\mathbb{k}G \otimes_{\mathbb{k}H} V = V \oplus g_2 V \oplus \dots \oplus g_r V.$$

In particular, we have that $\dim_{\mathbb{k}} \text{Ind}(V) = [G : H] \dim(V)$.

We can use this to give a more concrete definition of the G -action on induced modules. For each coset $g_i H$ of H and each g in G , we have that $g g_i H = g_j H$ for some $j \in [r]$, and this defines for each g in G a permutation σ_g of $[r]$. Thus, we can in general write $g g_i = g_{\sigma_g(i)} \mu_i(g)$ where $\mu_i(g) \in H$. This allows us to define an action of G on the direct sum $V \oplus g_2 V \oplus \dots \oplus g_r V$ as follows:

$$g \cdot (v_1, \dots, v_r) = \sigma_g \cdot (\mu_1(g)v_1, \dots, \mu_r(g)v_r).$$

Notice that since each $\mu_i(g) \in H$, the actions make sense, and that σ_g reorders the factors so that they land in the correct summand. We record this as a lemma:

Lemma 2. *A W is a G -module is induced from an H -module if and only if it contains an H -submodule V for which $W = s_1V \oplus \cdots \oplus s_rV$ for some cross-section of G with respect to H . In this case W is (isomorphic to) the induced module $\text{Ind}_H^G(V)$.*

The following follows from the usual tensor-hom adjunction, although it was originally proved “by hand” by Frobenius, using the description of the induced representation we just gave, which was for him a definition.

Lemma 3 (Frobenius reciprocity). *For any $\phi : G \longrightarrow H$, we have a natural isomorphism*

$$\text{hom}_H(\text{Ind}(V), W) \longrightarrow \text{hom}_G(V, \text{Res}(W))$$

In particular, in the semi-simple case, we have that $(\chi_{\text{Ind}V}, \chi_W)_G = (\chi_V, \chi_{\text{Res}W})_H$.

To understand why this is useful, notice that $\chi_{\text{Res}W}$ is trivial to compute from χ_W , while the computation of $\chi_{\text{Ind}V}$ is a bit more complicated. Let us explain how to carry it out.

If χ is a character of H (or, more generally, a class function) define a class function $\dot{\chi}$ on G by declaring that $\dot{\chi}(g) = 0$ if $g \notin H$, and which coincides with χ on H . Then, define for any $g \in G$

$$\text{Ind}(\chi)(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(x^{-1}gx).$$

Lemma 4 (The induced character formula). *If V is a representation of H , then $\chi_{\text{Ind}(V)} = \text{Ind}(\chi_V)$.*

Proof. This is **Exercise 4** of the corresponding exercise sheet. □

Example: dihedral groups. Let D_n be the dihedral group of order $2n$ generated by a rotation r and a reflection s . It contains a subgroup isomorphic to C_n and generated by r , whose irreducible complex representations are one dimensional, given by $\mathbb{C}$ where r acts by multiplication by ω^i for some primitive n th root of 1 in $\mathbb{C}$. We can pick $\{1, s\}$ as coset representatives, and we see immediately that the map $D_n \longrightarrow S_2$ sends r to 1 and s to (12) . Thus, the matrices for r and s of the two dimensional representation $\text{Ind}(\mathbb{C})$ are given, respectively, by

$$\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can check these 2-dimensional representations are irreducible, for example because these two matrices have no common eigenvectors. However, out of the n resulting 2-dimensional representations, only about half of them give representatives for the isomorphism classes of 2-dimensional irreducible representations. More precisely, one can show that the number of irreducible 2-dimensional representations of D_n is $\frac{n-2}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd.

Mackey's Theorems. If H and K are subgroups of G , one can induce a representation from H to G and then restrict it to K , effectively transferring information from H to K : a priori, there may be no group homomorphism in either direction from H to K . We now give some results of G. Mackey (see [here](#) for an account of his work in representation theory and physics) that describe how restriction induction and tensor products behave in certain situations, and which give a necessary and sufficient criterion for an induced representation to be irreducible.

Thus, let H and K be two subgroups of G , and write G as the disjoint union of double cosets HsK for some representatives $S \subseteq G$. If W is an H -module, we want to study the resulting K -module $\text{Res}_K^G \text{Ind}_H^G(W)$. For each $s \in S$, consider the subgroup $H_s = {}^sH \cap K$ of K where ${}^sH = sHs^{-1}$. If $x \in H_s$ then $sxs^{-1} \in H$, so H_s acts on W and produces a representation which we will write W_s .

Theorem 1 (Mackey decomposition). *The K -module $\text{Res}_K^G \text{Ind}_H^G(W)$ is isomorphic with*

$$\bigoplus_{s \in S} \text{Ind}_{H_s}^K(W_s).$$

More precisely, if we let Δ be the set of (K, H) -double cosets in G then we have a direct sum decomposition into K -submodules

$$\text{Res}_K^G \text{Ind}_H^G(W) = \bigoplus_{D \in \Delta} DW$$

and DW is isomorphic with $\text{Ind}_{H_g}^K(gW)$ for any g in D , where H_g acts on gW by left multiplication.

Notice that the last action makes sense, for if $y = gxg^{-1} \in H_g$ then $ygW = gxW$ is contained in gW , and thus this representation is isomorphic with W_g . We begin with a lemma:

Lemma 5 (Double Coset Lemma). *A subset $\{k_1, \dots, k_q\}$ is a cross section for K with respect to $H_g = K \cap {}^gH$ if and only if $\{k_1g, \dots, k_qg\}$ is a set of representatives for the left cosets of H contained in KgH .*

Proof. If $k_1, k_2 \in H$ then $k_1gH = k_2gH$ if and only if $k_1g = k_2gh$ for some $h \in H$, if and only if $k_2^{-1}k_1 = ghg^{-1} \in H_g$, if and only if $k_1H_g = k_2H_g$. $\square$

Proof of Mackey's decomposition theorem. Let $\{s_1, \dots, s_r\}$ be a cross section of H in G so that we have an H -module decomposition $\text{Ind}_H^G(W) = s_1W \oplus \dots \oplus s_rW$. If $g \in G$ then there exist $s_{i_1}, \dots, s_{i_l}$ for which KgH is the disjoint union of the cosets $s_{i_j}H$. Thus, DW is the direct sum of the corresponding $s_{i_j}W$, and it follows that we can group the summands above according to $D \in \Delta$, giving the desired K -decomposition. Thus, it suffices that we show DW is isomorphic with $(gW)_{H_g}^K$ for any $g \in D$.

First, DW contains gW as an H_g -submodule. Using the *Double Cosets Lemma*, we can assume that the s_{i_j} are of the form $\{k_1g, \dots, k_u g\}$ for some $k_i \in K$, so we have a decomposition

$$DW = k_1(gW) \oplus \cdots \oplus k_u(gW)$$

which shows that DW is induced from H_g from gW , as we wanted. $\square$

The following theorem tells us how induction behaves with the product of representations.

Theorem 2 (Mackey). *Let H_1 and H_2 be subgroups of G along with representations U_1 and U_2 , and let R be a set of representatives in G of the (H_1, H_2) -double cosets of G . For each $g \in R$ put $K_g = H_1 \cap {}^g H_2$. Then we have the following G -module decomposition:*

$$\text{Ind}_{H_1}^G(U_1) \otimes \text{Ind}_{H_2}^G(U_2) \cong \sum_{g \in R} \text{Ind}_{K_g}^G(\text{Res}_{K_g}(U_1) \otimes \text{Res}_{K_g}(gU_2)).$$

Notice that while U_1 is an H_1 -module and restriction happens through the inclusion $K_g \subseteq H_1$, the space gU_2 is a ${}^g H_2$ module and the restriction happens through the inclusion $K_g \subseteq {}^g H_2$.

Proof. The push-pull formula says that $\text{Ind}_{H_1}^G(-) \otimes -$ is isomorphic with $\text{Ind}_{H_1}^G(- \otimes \text{Res}_{H_1}^G(-))$, so we can begin by studying the module $\text{Res}_{H_1}^G(\text{Ind}_{H_2}^G(U_2))$. By Mackey's theorem, this is isomorphic to a sum of H_1 -modules isomorphic with $\text{Ind}_{K_g}^{H_1}(gU_2)$ for $g \in R$. Using the push-pull formula once again, we get

$$U_1 \otimes \text{Ind}_{K_g}^{H_1}(gU_2) = \text{Ind}_{K_g}^{H_1}(\text{Res}_{K_g}(U_1) \otimes \text{Res}_{K_g}(gU_2))$$

and, since induction is transitive, we can replace $\text{Ind}_{H_1}^G \text{Ind}_{K_g}^{H_1}$ with $\text{Ind}_{K_g}^G$. $\square$

Finally, suppose that W is a representation of H , and we consider $V = \text{Ind}(W)$. If W is reducible, then clearly V is reducible, but it may happen that V is reducible even if W is irreducible. For $s \in G$ and $H_s = sHs^{-1} \cap H$, we now have two different ways of getting an H_s -module out of W : we can either consider the conjugate representation W_s that we introduced before, and we can restrict the action of H to that of H_s , which we write W^s . We now assume we are working over an algebraically closed field for which the corresponding group algebra is semi-simple, although one can state the following in a more general situation:

Theorem 3 (Mackey's Criterion). *For $\text{Ind}(W)$ to be irreducible, it is necessary and sufficient that W be irreducible, and that for each $s \notin H$, the H_s -modules W_s and W^s are orthogonal.*

We say two modules are orthogonal if the irreducible representations appearing in their decomposition are non-isomorphic. In other words, their characters are orthogonal class functions.

Proof. As before, write $V = \text{Ind}(W)$. We compute (χ_V, χ_V) . By the Frobenius reciprocity theorem, we have that $(\chi_V, \chi_V) = (\chi_W, \text{Res}(\chi_V))$, and by Mackey's Decomposition Theorem we have that

$$\text{Res}(V) = \bigoplus_{s \in S} \text{Ind}_{H_s}^H(W_s).$$

If we apply the reciprocity theorem again we get that

$$(\chi_V, \chi_V) = \sum_{s \in S} (\chi_{W^s}, \chi_{W_s}).$$

For $s = 1$ we get $W^1 = W_1 = W$ so the summand $(\chi_W, \chi_W) \geq 1$. Hence, that inner product is 1 if and only if W is irreducible and all other inner products are zero, like we wanted. $\square$

Corollary 1. *If H is normal, then $\text{Ind}(W)$ is irreducible if and only if W is irreducible and not isomorphic to any of its conjugate representations.*

Recall that if H is normal in G and V is an H -module, then for each $g \in G$ the H -module ${}^gV$ with action $h \cdot v = ghg^{-1}v$ is called a conjugate of V .