

Representation Theory — WiSe 2022

Notes for Lecture 7

Following the notation from **Lecture 6**, we write F_D for the Young symmetriser for a diagram D of shape μ . During that lecture, we proved that if μ is larger than λ lexicographically, then $S_\mu x A_\lambda = 0$ for any $x \in \mathbb{Q}S_n$.

Induced representations. For μ a partition of n and D any diagram of shape μ , consider the row-stabilizer subgroup $S_\mu = R(D)$. We know that S_μ is unique up to conjugation in S_n , so the notation is not ambiguous, and write U_μ for the representation of S_n induced from the trivial representation.

Notice that if H is a subgroup of G and if $u = \sum_{h \in H} h$, then $\mathbb{K}[G]u$ is isomorphic to the representation of G induced from the trivial representation of H . In particular, U_μ is isomorphic to the S_n -module $\mathbb{K}[S_n]S_D$ where D is any diagram with shape μ .

Proposition 1. *If $\mu < \lambda$ then $\text{Hom}_{S_n}(U_\lambda, V_\mu) = 0$, while $\text{Hom}_{S_n}(U_\lambda, V_\lambda) = \mathbb{Q}$. In other words, there exist integers $K'_{\lambda\mu}$ which give an upper triangular decomposition:*

$$U_\lambda = V_\lambda + \sum_{\mu > \lambda} K'_{\lambda\mu} V_\mu.$$

The proposition follows immediately from the lemma above. Hence, the idea to keep in mind is that we can compute the character of U_λ using the induced character formula, and then invert the linear system to compute the character of the Specht modules. To compute the character of U_λ , we introduce some polynomials.

Definition 1. *The power-sum symmetric polynomial $p_m(x)$ on variables $x = (x_1, \dots, x_N)$ is*

$$p_m(x) = x_1^m + \dots + x_N^m.$$

It is symmetric in the sense it is S_N -invariant. Given a cycle type $\mathbf{i} = (i_1, i_2, \dots)$ in S_n with corresponding partition μ , we define:

$$p_\mu(x) = p_{\mu_1}(x) p_{\mu_2}(x) p_{\mu_3}(x) \cdots = p_1(x)^{i_1} p_2(x)^{i_2} p_3(x)^{i_3} \cdots$$

where x contains at least as many variables as parts in λ . Let us write ψ_λ for the character of U_λ .

Theorem 1. We have that $\psi_\lambda(C_\mu)$ is the coefficient of x^λ in $p_\mu(x)$ as long as x contains n or more variables.

Proof. According to the formula for the induced character, we have to count the number of elements $\sigma \in S_n$ such that $\sigma^{-1}\tau\sigma \in R(\lambda)$ where τ is any representative of the conjugacy class for μ , and divide this number by $|R(\lambda)| = \lambda! = \lambda_1!\lambda_2!\cdots$.

The first number is equal to the number of elements in $R(\lambda)$ conjugate to τ , which is $|C_\mu \cap R(\lambda)|$, times the order of the centralizer of τ , which is $n!/|C_\mu|$ by the orbit-stabilizer theorem. All in all, the number we are after is

$$\frac{n!}{\lambda!} \frac{|C_\mu \cap R(\lambda)|}{|C_\mu|},$$

which we proceed to simplify. First, we know that $n!/|C_\mu| = 1^{i_1}i_1!2^{i_2}i_2!\cdots$, so it suffices that we compute $|C_\mu \cap R(\lambda)|$. We can compute this number by noting that, by looking at each component of $R(\lambda)$, that permutes λ_j components, we must be able to decompose it into a product of r_{jm} m -cycles for $m \geq 1$ in such a way that:

$$r_{j1} + 2r_{j2} + \cdots = \lambda_j, \quad r_{1m} + r_{2m} + \cdots = i_m.$$

By the same formula we used to compute $|C_\mu|$, we get that

$$|C_\mu \cap R(\lambda)| = \sum_{\mathbf{r}} \frac{\lambda!}{\prod_{m,j} m^{r_{jm}} r_{jm}!}.$$

To sum up, after some cancellation, we get that the coefficient is

$$\psi_\lambda(C_\mu) = \sum_{\mathbf{r}} \prod_m \frac{i_m!}{\prod_j r_{jm}!}$$

and it suffices to show this is the coefficient of x^λ in $p_\mu(x) = \prod_m (x_1^m + \cdots + x_N^m)^{i_m}$. Suppose that when expanding this product, we pick the monomial x_j^m exactly r_{jm} times and we get x^λ . Then, we must have that $\sum_j r_{jm} = i_m$, and that $\sum_m m r_{jm} = \lambda_j$. It suffices to notice that, due to symmetry of the factors, we can do this in $i_m! / \prod_j r_{jm}!$ ways for each choice of r , which is what we wanted. $\square$

Before giving a proof of the formula for the character of irreducible representations, we introduce symmetric functions.

Symmetric functions. The symmetric group S_n acts on $\mathbb{Z}[x_1, \dots, x_n]$, and we write Λ_n for the ring of S_n -invariant polynomials, which we call *symmetric*. For any partition λ of length at most n , we have a symmetric polynomial m_λ obtained by summing over all permutations of x^λ . By their very definition, we have that:

Lemma 1. *The polynomials m_λ such that $\ell(\lambda) \leq n$ form a $\mathbb{Z}$ -basis of Λ_n , and specializing to those $\lambda \vdash k$ gives a basis of $\Lambda_n^{(k)}$, those symmetric polynomials on n variables that are homogeneous of degree k . In particular, the m_λ that form a partition of k are a basis for $\Lambda_n^{(k)}$ as soon as $n \geq k$. $\square$*

Usually, it will be more convenient to work over an infinite number of variables, since most formulas we will obtain stabilize as soon as the number of variables is large enough. To be more precise, whenever $m \geq n$ there is a ring homomorphism $\Lambda_m \rightarrow \Lambda_n$ that sends $x_i \rightarrow x_i$ if $i \in [n]$ and send $x_{n+1}, \dots, x_m$ to zero. It has the property that $m_\lambda(x_1, \dots, x_m)$ goes to $m_\lambda(x_1, \dots, x_n)$ if λ has at most n parts, and to zero else. A symmetric function (in infinitely many variables) is a sequence of elements $f_n \in \Lambda_n$ such that $\pi_n^m(f_m) = f_n$. We take this point of view and explain how such formulas specialize when it is convenient.

The *elementary symmetric function* e_r is given by $m_{(1^r)}$, which is simply the product of all possible distinct monomials in r different variables. From its definition, it follows that

$$E(t) = \sum_{r \geq 1} e_r t^r = \prod_{i \geq 1} (1 + x_i t).$$

If the number of variables is finite, then both product and sum stop at n . For each partition λ of n , we define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots$. Finally, we define the *natural* ordering on partitions as follows: we say that $\mu \leq \lambda$ if for all i we have that $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$.

Lemma 2. *Let $\alpha \in \mathbb{Z}^n$. Then $\alpha_1 \geq \alpha_2 \geq \dots$ if and only if $\alpha \geq \sigma \alpha$ for any permutation $\sigma \in S_n$.*

That is, the unique permutation of α that is a partition is the largest element in the orbit of α with respect to this natural order.

Lemma 3. *The polynomials $e_1, e_2, \dots$ are algebraically independent over $\mathbb{Z}$ and $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$. More precisely, we have that*

$$e_{\lambda'} = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$$

where the $a_{\lambda\mu}$ are non-negative integers, λ' is the conjugate partition to λ , and the sum is taking over those μ smaller than λ in the natural order.

Proof. When we multiply out $e_{\lambda'}$, we obtain a sum of monomials of the form

$$(x_{i_1} x_{i_2} \dots)(x_{j_1} x_{j_2} \dots) \dots = x^\alpha$$

where $i_1 < \dots < i_{\lambda'_1}$, $j_1 < \dots < j_{\lambda'_2}$, and so on. Let us now fill in the diagram for λ using the i s in the first column (in order), the j s in the second column, and so on. For each $r \geq 1$, all the numbers $\leq r$ entered in the diagram must occur in the first r rows, as the variables are all distinct and strictly increasing. It follows that $\alpha_1 \leq \lambda_1$, as this is the number of 1s entered. Similarly,

$\alpha_1 + \dots + \alpha_i \leq \lambda_1 + \dots + \lambda_i$. By Lemma 2, there is one of these α which is a partition, and all other are smaller, so we can write

$$e_{\lambda'} = m_{\lambda} + \sum_{\mu < \lambda} a_{\lambda\mu} m_{\mu}$$

where the coefficient $a_{\lambda\lambda} = 1$ is justified, as there is only one way to obtain x^{λ} as a product above, namely: all 1s must lie on the first row, all 2s in the second, and so on. $\square$

The *complete symmetric function* h_r is defined as the sum $h_r = \sum_{|\lambda|=r} m_{\lambda}$, so that

$$H(t) = \sum_{r \geq 1} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}.$$

Proposition 2. *We have that $H(t)E(-t) = 1$. In other words, for all $n \geq 1$ we have that*

$$\sum_{r=0}^n (-1)^r e_r h_{n-r} = 0.$$

Since the e_r are algebraically independent, we can define a self-map ω on Λ sending $e_r \mapsto h_r$. The equation above implies that $\omega^2 = 1$, so that the h_r are also algebraically independent and generate Λ as a ring. We define h_{λ} as before, which gives us another basis of Λ .

Power sums. We already have seen the power sum symmetric polynomials, which are just $m_{(r)}$. The generating function is

$$P(t) = \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t} = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}.$$

Similarly, $P(-t) = E'(t)/E(t)$, which gives us a recursion to compute h and e in terms of themselves and p , at the cost of dividing by integers:

$$nh_n = \sum_{r=1}^n p_r h_{n-r}, \quad ne_n = \sum_{r=1}^n (-1)^{r-1} p_r e_{n-r}.$$

Lemma 4. *The polynomials $p_1, p_2, \dots$ are algebraically independent over $\mathbb{Z}$, and they span $\Lambda_{\mathbb{Q}}$. Moreover, if we put $z_{\lambda} = 1^{i_1} i_1! 2^{i_2} i_2! \dots$, we have that*

$$\begin{aligned} h_n &= \sum_{|\lambda|=n} z_{\lambda}^{-1} p_{\lambda}, \\ e_n &= \sum_{|\lambda|=n} \varepsilon_{\lambda} z_{\lambda}^{-1} p_{\lambda} \end{aligned}$$

where $\varepsilon_{\lambda} = (-1)^{|\lambda| - \ell(\lambda)}$.

Proof. The first statement is clear from Newton's formulas. Since $\omega(p_n) = (-1)^{1-n}p_n$ and ω interchanges e and h , it suffices that we prove the first formula. To do this, we integrate $P = H'/H$ formally and obtain:

$$\begin{aligned} H(t) &= \exp \left(\sum_{r \geq 1} p_r \frac{t^r}{r} \right) \\ &= \prod_{r \geq 1} \exp \left(p_r \frac{t^r}{r} \right) \\ &= \prod_{r \geq 1} \sum_{m_r \geq 0} \frac{p_r^{m_r} t^{rm_r}}{r^{m_r} m_r!} \\ &= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda} t^{|\lambda|}. \end{aligned}$$

where the last equality follows by passing from the 'cycle type' notation of a partition to the usual notation, by collecting terms according to the value of $\sum_{r \geq 1} rm_r$. $\square$

Schur functions. Let us consider a monomial x^{α} in a finite number n of variables, and let us anti-symmetrize it into

$$a_{\alpha} = \sum_{\sigma} (-1)^{\sigma} \sigma(x^{\alpha}).$$

Since a_{α} is now skew-symmetric, it vanishes unless the entries of α are all distinct, so we may assume they are strictly increasing. In this case, there exists a partition λ for which $\alpha = \lambda + \delta$ where $\delta = (n-1, n-2, \dots, 1, 0)$. In this case, we have in fact written $a_{\lambda+\delta}$ as the determinant of the matrix with $A_{ij} = x_i^{\lambda_j+n-j}$, and this is divisible by a_{δ} , the Vandermonde determinant, as it is the case for any skew polynomial.

Definition 2. We define $s_{\lambda} = a_{\lambda+\delta}/a_{\delta}$ and call it the Schur function associated to λ in the variables $x_1, \dots, x_N$.

Since the $a_{\lambda+\delta}$ form a basis of the alternating polynomials and multiplication by a_{δ} gives a bijection between skew and symmetric polynomials, we get that:

Lemma 5. The Schur functions s_{λ} on n variables associated to partitions λ with at most n parts form an integer basis for the symmetric polynomials in n variables.

The following *Jacobi–Trudi formulas* express the Schur polynomials in terms of polynomials in the h and the e bases. We omit their proof and point the reader to [1, 2].

Proposition 3. For any partition λ , we have that s_{λ} is the determinant of the matrix A with $A_{ij} = h_{\lambda_i - i + j}$, and of the matrix B with $B_{ij} = h'_{\lambda'_i - i + j}$. $\square$

Orthogonality relations. To relate symmetric functions to characters of the symmetric groups, we now prove analogous orthogonality relations in Λ . We consider the infinite product

$$R(x, y) = \prod_{i, j \geq 1} \frac{1}{1 - x_i y_j}$$

over two countable sets of variables.

Lemma 6. *We have that $R(x, y) = \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) p_{\lambda}(y)$.*

Proof. Use Lemma 4 to expand $\prod_{j \geq 1} H(y_j)$ in terms of the power-sum functions. Concretely, note that

$$\begin{aligned} \prod_{j \geq 1} H(y_j) &= \prod_{j \geq 1} \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) y_j^{|\lambda|} \\ &= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) p_{\lambda}(y) \end{aligned}$$

by the very definition of the power-sum symmetric functions. □

Lemma 7. *We have that $R(x, y) = \sum_{\lambda} h_{\lambda}(x) m_{\lambda}(y) = \sum_{\lambda} m_{\lambda}(x) h_{\lambda}(y)$.*

Proof. We again expand $\prod_{j \geq 1} H(y_j)$ but this time we write

$$\begin{aligned} \prod_{j \geq 1} H(y_j) &= \prod_{j \geq 1} \sum_{r \geq 0} h_r(x) y_j^r \\ &= \prod_{\alpha} h_{\alpha}(x) y^{\alpha} \\ &= \prod_{\lambda} h_{\lambda}(x) m_{\lambda}(y). \end{aligned}$$

The second equality follows since R is symmetric in x and y . □

Finally, we have the following crucial result, which requires a slightly more technical lemma, so we omit its proof. For details, see (4.3) in [1] or 7.12.1 in [2] for a proof using the Robinson—Schensted—Knuth algorithm.

Lemma 8. *We have that $R(x, y) = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)$ and $\omega(s_{\lambda}) = s_{\lambda'}$.* □

Let us define a $\mathbb{Z}$ -valued bilinear form in Λ by declaring that m and h are dual to each other, that is

$$\langle h_{\mu}, m_{\lambda} \rangle = \delta_{\mu, \lambda}.$$

The following result implies that Schur functions are orthonormal for this product:

Proposition 4. *Consider $\mathbb{Q}$ -bases u and v of $\Lambda_{\mathbb{Q}}$ indexed by partitions. Then $\langle u_{\mu}, v_{\lambda} \rangle = \delta_{\mu, \lambda}$ for all μ and λ if and only if $R(x, y) = \sum_{\lambda} u_{\lambda}(x) v_{\lambda}(y)$.*

Proof. If A is the transition matrix for u in base h , and if B is the transition matrix for v in base m , then the orthogonality condition is equivalent to $AB^t = 1$. On the other hand, the functional equation is equivalent to

$$\sum_{\lambda} u_{\lambda}(x) v_{\lambda}(y) = \sum_{\lambda} h_{\lambda}(x) m_{\lambda}(y)$$

which is equivalent to $A^t B = 1$. Thus, all statements are equivalent. $\square$

This implies, in particular, that up to ordering and sign, the Schur functions are the unique orthonormal basis of Λ , since any orthogonal integer matrix must be a signed permutation matrix. Moreover, we deduce that our bilinear form is symmetric and positive definite, and that ω preserves the product, since $\langle p_{\mu}, p_{\lambda} \rangle = \delta_{\mu\lambda} z_{\mu}$ and ω multiplies the p with a sign.

Transition matrices. In this section we order partitions using the reverse lexicographical order, so that (1^n) is the largest partition of $n \in \mathbb{N}$ and (n) is the smallest. In particular, if λ is greater than μ in the natural order, then μ is larger than λ . We also mention without proof that the natural order $\geq$ has the property that

$$\lambda \geq \mu \iff \mu' \geq \lambda'.$$

We say a matrix indexed by the partitions of n is strictly upper triangular if $M_{\lambda\mu} = 0$ unless $\lambda \geq \mu$ in the natural order. We say it is strictly upper uni-triangular if its diagonal entries are all 1. It is not hard to check that the groups of strictly upper (resp. lower) uni-triangular matrices form a group.

Lemma 9. *Let J be the matrix with $J_{\mu\lambda} = 1$ if and only if $\mu = \lambda'$. Then $J^2 = 1$ and $M \mapsto JMJ$ is an involution exchanging strictly upper (uni-)triangular and strictly lower (uni-)triangular matrices.*

Given $\mathbb{Q}$ -bases u and v of Λ^n , indexes by partitions of n , we denote $M = M(u, v)$ the unique matrix for which

$$u_{\lambda} = \sum_{\mu} M_{\lambda\mu} v_{\mu}$$

and call it the transition matrix from u to v . As usual, we have that

- (i) $M(u, v)M(v, w) = M(u, w)$
- (ii) $M(u, v)^{-1} = M(v, u)$
- (iii) $M(u', v') = M(v, u)^t = M(u, v)^*$

where u' and v' are the dual bases with respect to our scalar product, and M^* denotes the transpose of the inverse of M . Moreover, $M(u, v) = M(\omega u, \omega v)$.

Proposition 5. *Let $K = M(s, m)$ be the transition matrix expressing the Schur functions in terms of the monomial functions, and let J be the involutive matrix defined above. Then all transition matrices involving the bases e, h, m and s can be expressed in terms of J and K .*

Proof. To begin with, since h and m are dual, and since s is self-dual, we get that $M(s, h) = K^*$. Since $J = M(\omega(s), s)$, we get $M(s, e) = M(\omega(s), h) = M(\omega(s), s)M(s, h) = JK^*$. This is enough to compute the remaining matrices. $\square$

We will soon see that the entries of K , which we call *Kostka* numbers, are the entries of the resulting matrix K' introduced in Proposition 1. We point out, without proof, that one can nonetheless already show that K is a matrix of non-negative integers, and that it is strictly upper uni-triangular, by giving a combinatorial interpretation for such numbers. We content ourselves with the following example, where we say that an integer matrix A of size $m \times n$ has *column content* $\alpha \in \mathbb{Z}^n$ if for each $i \in [n]$, the integer α_i is equal to the sum of the entries of the i th column of A . Similarly, we speak of the *row content* of an (integer) matrix.

Proposition 6. *For any two partitions λ and μ , we have that $M(e, m)_{\lambda\mu}$ is the number of $(0, 1)$ -matrices with row content λ and column content μ , while $M(h, m)$ is the number of non-negative integer matrices with row content λ and column content μ .*

Proof. Consider the coefficient of x^μ in e_λ . Each monomial in e_{λ_i} is of the form $\prod_j x_j^{a_{ij}}$ where the a_{ij} are 0 or 1 and $\sum_j a_{ij} = \lambda_i$. To get x^μ after multiplication, we must have

$$\prod_{i,j} x_j^{a_{ij}} = \prod_j x_j^{\mu_j}$$

which says that $\sum_i a_{ij} = \mu_j$. The case of h is similar, the only difference is that a_{ij} can now be any non-negative integer. $\square$

The character formula and the character ring. For each $n \geq 1$, let us write R_n for the ring of characters of S_n : we know all characters of S_n are integer valued, so their products and $\mathbb{Z}$ -linear combinations form a ring. We agree that $R_0 = \mathbb{Z}$, and then put

$$R = \bigoplus_{n \geq 0} R_n.$$

This is a priori a graded Abelian group whose homogeneous components are rings, but we can make it into a ring, as follows:

Definition 3. *If $f \in S_n$ and $g \in S_m$, we define $f \cdot g$ to be the induced character $\text{Ind}_{S_n \times S_m}^{S_{n+m}}(f \otimes g)$ where $f \otimes g$ is the character of $S_n \times S_m$ on the corresponding tensor product.*

Since $S_n \times S_m$ is well-defined up to conjugation in S_{m+n} , our formula is also well-defined. Notice also that we can define a map $\times : S_n \times S_m \rightarrow S_{m+n}$, well-defined up to conjugacy. In turn, let us define a map $\Phi : S_n \rightarrow \Lambda^{(n)}$ for which $\Phi(\sigma) = p_{\lambda(\sigma)}$, where $\lambda(\sigma)$ is the cycle type of σ . It follows that Φ is multiplicative, in the sense that $\Phi(\sigma \times \tau) = \Phi(\sigma)\Phi(\tau)$. Notice that Φ is, by its very definition, a class function.

Definition 4. The Frobenius character map $\text{ch} : R \longrightarrow \Lambda_{\mathbb{C}}$ is defined on $f \in R_n$ by

$$\text{ch}(f) = \langle f, \Phi \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \Phi(\sigma).$$

Notice that collecting terms into cycle types and using the usual formula for the size of a conjugacy class, we obtain that

$$\text{ch}(f) = \sum_{|\lambda|=n} z_{\lambda}^{-1} f_{\lambda} p_{\lambda}$$

where $f_{\lambda} = f(C_{\lambda})$. The main result about ch is that it provides us with an isometric isomorphism between R and $\Lambda_{\mathbb{C}}$.

Theorem 2. The Frobenius character map is an isometric ring isomorphism.

Proof. That ch is an isometry follows from Proposition 6 and Proposition 4. That it is an algebra morphism follows from the Frobenius reciprocity theorem and the previous observation that Φ is multiplicative for the embedding of $S_n \times S_m$ into S_{m+n} . Namely

$$\begin{aligned} \text{ch}(f \cdot g) &= \langle \text{Ind}_{S_m \times S_n}^{S_{m+n}} (f \otimes g), \Phi \rangle_{S_{m+n}} \\ &= \langle f \otimes g, \text{Res}_{S_m \times S_n}^{S_{m+n}} \Phi \rangle_{S_m \times S_n} \\ &= \langle f, \Phi \rangle_{S_m} \langle g, \Phi \rangle_{S_n}. \end{aligned}$$

Finally, it suffices to show that ch is surjective, since ch is already isometric and the bilinear forms are non-degenerate. First, if 1_n is the trivial representation of S_n then we see that $\text{ch}(1_n) = h_n$ by Lemma 4. Thus, if ψ_{λ} is the character of U_{λ} , we see that $\text{ch}(\psi_{\lambda}) = h_{\lambda}$, so that ch is surjective. $\square$

Notice that the ψ_{λ} form a basis for R_n , since we have exhibited them as a triangular change of basis from the irreducible characters χ_{λ} in Proposition 1.

Theorem 3 (Character formula). The transition matrix $M(p, s)$ is the character table of S_n . That is, for each partition μ of n we have that

$$p_{\mu} = \sum_{\lambda \vdash n} \chi_{\lambda}(C_{\mu}) s_{\lambda}.$$

In particular, $\chi_{\lambda}(C_{\mu})$ is equal to the coefficient of $x^{\lambda+\delta}$ in $a_{\delta} p_{\mu}$.

The last statement of this theorem is known as the *Frobenius character formula*. Recall that a virtual character is an *integer* linear combination of characters. If φ is a virtual character such that $(\varphi, \varphi) = 1$, then φ or $-\varphi$ is the character of an irreducible representation, depending on whether $\varphi(1) > 0$ or $\varphi(1) < 0$.

Proof. Using the Jacobi–Trudi formula and the fact that $\text{ch}(1_n) = h_n$, we can construct a *virtual* character χ'_λ of S_n defined by $\chi'_\lambda = \det(1_{\lambda_i - i + j})_{i,j}$, which then must have the property that

$$\text{ch}(\chi'_\lambda) = s_\lambda.$$

This implies that these virtual characters are orthonormal, and thus to check that they correspond to an actual character, we just need to check that $\chi'_\lambda(1) > 0$. By the way we have defined them, we see that

$$s_\lambda = \text{ch}(\chi'_\lambda) = \sum_{\mu \vdash n} z_\mu^{-1} \chi'_\lambda(C_\mu) p_\mu$$

so we deduce that $\langle s_\lambda, p_\mu \rangle = \chi'_\lambda(C_\mu)$. In particular $\chi'_\lambda(1) = \langle s_\lambda, p_1^n \rangle$ or, what is the same

$$p_1^n = \sum_{\lambda \vdash n} \chi'_\lambda(1) s_\lambda.$$

We conclude that $\chi'_\lambda(1) = M(h, s)_{\lambda, (1^n)} = K_{\lambda, (1^n)} > 0$. Finally, we have to show that if χ_λ is the character of the Specht module V_λ , then $\text{ch}(\chi_\lambda) = s_\lambda$, which shows that $\chi_\lambda = \chi'_\lambda$ and concludes our proof.

We know at least that if we write $\text{ch}(\chi_\lambda) = t_\lambda$ then the t are orthonormal symmetric polynomials coming from a character so, up to a permutation of partitions, they must be equal to the Schur polynomials. We also know that

$$h_\lambda = t_\lambda + \sum_{\mu > \lambda} K'_{\lambda\mu} t_\mu.$$

On the other hand, by definition we have that

$$h_\lambda = s_\lambda + \sum_{\mu > \lambda} K_{\lambda\mu} s_\mu.$$

Proceeding by induction using the fact that s and t are both bases, we conclude both that $s = t$ and that $K = K'$, which completes the proof of the theorem. $\square$

References

- [1] Ian G. Macdonald, *Symmetric functions and hall polynomials*, Oxford University Press, 2015.
- [2] Richard P. Stanley and Sergey Fomin, *Enumerative combinatorics*, Cambridge University Press, 1999.