

# Representation Theory — WiSe 2022

## *Simple representations, Schur's Lemma and characters*

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1. Let  $G$  be a finite group and let  $G' = [G, G]$  be its derived subgroup. Let  $G^{\text{ab}} = G/G'$  be the Abelianization of  $G$ .

- (i) If  $V$  is a one-dimensional representation of  $G$ , show there exists a morphism of groups  $\chi_V : G \longrightarrow \mathbb{k}^\times$  such that  $gv = \chi_V(g)v$  for all  $g \in G$  and  $v \in V$ .
- (ii) Show that if  $V'$  is another one dimensional representation,  $V$  and  $V'$  are isomorphic if and only if  $\chi_V = \chi_{V'}$ .
- (iii) Conversely, show that if  $\psi : G \longrightarrow \mathbb{k}^\times$  is a morphism then there is  $V$  a one dimensional representation with  $\chi_V = \psi$ .
- (iv) Prove that there are exactly  $|\text{hom}(G^{\text{ab}}, \mathbb{k}^\times)|$  one dimensional representations.
- (v) For each  $n \geq 2$ , determine all one dimensional representations of  $S_n$ .

1. *Solution.* To prove the first point, take a one dimensional representation  $\rho : G \longrightarrow \text{GL}(V)$  and pick any non-zero  $v \in V$ , so that  $V = \langle v \rangle$ .

For each  $g \in G$ , there exists  $\chi_V(g) \in \mathbb{k}^\times$  such that  $gv = \chi_V(g)v$ , and if we had chosen another  $v'$ , then  $v' = \lambda v$  for some scalar so  $gv' = \lambda gv = \lambda \chi_V(g)v = \chi_V(g)v'$ . Hence,  $\chi_V(g)$  depends only on  $g$  and  $V$ , and defines a map  $G \longrightarrow \mathbb{k}^\times$ . This is a morphism of groups, for  $g(hv) = g\chi_V(h)v = \chi_V(h)\chi_V(g)v$  while  $(gh)v = \chi_V(gh)v$ .

To prove the second point, suppose that  $\psi : V \longrightarrow V'$  is an isomorphism of one-dimensional representations, necessarily given by sending a basis (non-zero)  $v \in V$  to another non-zero  $v' \in V'$ . Then  $\psi(gv) = \psi(\chi_V(g)v) = \chi_V(g)v'$  while  $g\psi(v) = gv' = \chi_{V'}(g)v'$ , hence  $\chi_V = \chi_{V'}$ . Conversely, if we have this equality this computation shows that any non-zero map is an equivariant isomorphism. If  $\psi : G \longrightarrow \mathbb{k}^\times$  is a group homomorphism, define  $V = \mathbb{k}$  and  $g\lambda = \psi(g)\lambda$ . Then clearly  $\chi_V = \psi$ . It follows that one-dimensional representations are in bijection with maps  $G \longrightarrow \mathbb{k}^\times$  and, since  $\mathbb{k}^\times$  is Abelian, with maps  $G^{\text{ab}} \longrightarrow \mathbb{k}^\times$ .

For the last point, we know that  $S_n^{\text{ab}} = C_2$ , the cyclic group of order two. If  $1 \neq -1$  in  $\mathbb{k}$  then  $S_n$  has two irreducible representations, obtained from the trivial map  $S_n \rightarrow C_2$  and the non-trivial sign map  $S_n \rightarrow C_2$ . If not,  $S_n$  admits only the one dimensional trivial representation.

2. Let  $V$  be an  $S_3$ -module and consider the cyclic subgroup  $C_3 = \langle (123) \rangle \subseteq S_3$ .

- (i) Show that  $V$  decomposes into a direct sum  $V_0 \oplus V_1 \oplus V_2$  of eigenspaces for  $(123)$  with eigenvalues  $1, \omega$  and  $\omega^2$  for  $\omega$  a primitive 3rd root of unity.
- (ii) Show that if  $v \in V$  is an eigenvector of eigenvalue  $\omega^i$  for  $(123)$  and  $\tau$  is a transposition, then  $\tau(v)$  is an eigenvector for  $(123)$  of eigenvalue  $\omega^{2i}$ .
- (iii) Show that if  $v \in V_0$  then  $\langle v, \tau(v) \rangle$  is either one dimensional or decomposes as the sum of the trivial and the sign representation.
- (iv) Show that if  $v \in V_1$  or  $V_2$  then  $\langle v, \tau(v) \rangle$  is isomorphic to the standard representation.

**2. Solution.** Since  $(123)$  has order 3 in  $S_3$ , we see that  $X^3 - 1$  is the minimal polynomial of the corresponding linear transformation. It splits completely as  $(X - 1)(X - \omega)(X - \omega^2)$ , so the action is diagonalizable and hence  $V$  decomposes accordingly into the stated eigenspaces.

If  $\tau$  is any transposition in  $S_3$ , then  $(123)\tau = \tau(123)^2$ , so it follows that if  $v$  is an eigenvector of  $(123)$  with eigenvalue  $\omega^i$ , then  $\tau(v)$  is an eigenvector with eigenvalue  $\omega^{2i}$ :  $(123)\tau v = \tau(123)^2 v = \omega^{2i}\tau v$ . This proves the second item.

If  $\langle v, \tau v \rangle$  is one dimensional then  $\tau v = \lambda v$  for some  $\lambda$ , and  $\tau^2 v = v$  forces  $\lambda^2 = 1$  so  $\lambda = 1$  or  $-1$ . Thus, if  $\langle v, \tau v \rangle$  is not one dimensional,  $v - \tau(v)$  and  $v + \tau(v)$  are both non-zero, and are linearly independent. It is clear the first spans the sign representation and the second spans the trivial representation, which is what we are required to show.

Finally, we prove that if  $v \in V_1$  or  $V_2$  then  $\langle v, \tau(v) \rangle$  is isomorphic to the standard representation. Notice that the standard representation  $\mathbb{C}^2$  admits a basis  $u_1 = (1, \omega, \omega^2)$  and  $u_2 = \tau u_1$  with the property that  $(123)u_1 = (\omega, \omega^2, 1) = \omega u_1$  and  $(123)u_2 = \omega^2 u_2$  by what we have just proved. If  $v \in V_1$ , consider the morphism  $v \mapsto u_1, \tau(v) \mapsto u_2$  and if  $v \in V_2$ , we simply interchange the roles of  $u_1$  and  $u_2$ . This is bijective, so it suffices to show it is  $S_3$ -equivariant. Since  $\tau$  and  $(123)$  generate  $S_3$ , it suffices to prove that it is equivariant for these two elements. By construction, it is equivariant for  $\tau$ , and one can directly check that it is equivariant for  $(123)$ .

3. Let  $X$  be a  $G$ -set and form  $V = \mathbb{k}X$  where  $\mathbb{k}$  has zero characteristic.

- (i) Show that  $\dim V^G = |X/G|$ , the number of orbits of  $G$  on  $X$ .
- (ii) Show that  $V^G$  is one dimensional if and only if  $X$  is a transitive  $G$ -set.
- (iii) Compute  $\chi_V$  and  $\langle \chi_V, 1 \rangle$  using the inner product formula.
- (iv) Deduce Burnside's Lemma, which gives formula for the number of orbits of  $G$  on  $X$  in terms of fixed point sets.

**3. Solution.** Let  $v = \sum_{x \in X} \lambda_x e_x$ , and notice that  $gv = \sum_{x \in X} \lambda_x g e_x = \sum_{x \in X} \lambda_x e_{gx} = \sum_{y \in X} \lambda_{g^{-1}y} e_y$ , so the coefficient of  $e_x$  in  $gv$  is  $\lambda_{g^{-1}x}$ . Hence  $v$  is  $G$ -invariant if and only if for all  $x \in X$  and all  $g \in G$  we have that  $\lambda_{g^{-1}x} = \lambda_x$  or, what is the same, if  $\lambda_x = \lambda_y$  whenever  $x$  and  $y$  are in the same  $G$ -orbit. This proves that the elements  $e_{\mathcal{O}} = \sum_{g \in \mathcal{O}} g$  for  $\mathcal{O}$  ranging through the orbits of  $G$  in  $X$  span  $V^G$ , and they are linearly independent since the orbits are disjoint. From this we deduce that  $V^G$  is one dimensional if and only if  $X$  is a transitive  $G$ -set.

To compute  $\chi_V$ , we notice that the matrix of  $g$  for the basis  $\{e_x : x \in X\}$  has a 1 in the diagonal for each  $x \in X$  such that  $gx = x$ , and hence  $\chi_V(g) = \text{Fix}(g)$ , the number of fixed points of  $g$ . Thus, we can compute that

$$\langle \chi_V, 1 \rangle = \frac{1}{|G|} \sum_{g \in G} \text{Fix}(g)$$

is equal to the average number of fixed points of  $G$  on  $X$ . The second item then shows this is equal to  $|X/G|$ , giving Burnside's Lemma.

**4.** Continuing from the previous question, assume that  $G$  acts transitively on  $X$ , so that  $V = \mathbb{k} \oplus W$ . Let  $V'$  be the representation afforded by the diagonal action of  $G$  on  $X \times X$ .

- (i) Show that  $\langle \chi_V, \chi_V \rangle = 1 + \langle \chi_W, \chi_W \rangle$  and that  $\chi_V^2 = \chi_{V'}$ .
- (ii) Show that  $G$  has at least one orbit in  $X \times X$  given by the diagonal  $\Delta X = \{(x, x) : x \in X\}$ .
- (iii) Show that  $\langle \chi_V, \chi_V \rangle = \langle \chi_{V'}, 1 \rangle$  and conclude that  $G$  acts doubly transitively on  $X$  if and only if  $W$  is irreducible.

**4. Solution.** Let us assume that the action is transitive so that  $V = \mathbb{k} \oplus W$ . Since  $W$  does not contain a copy of the trivial representation,  $\langle 1, \chi_W \rangle = 0$ , so that from  $\chi_V = 1 + \chi_W$  we obtain  $\langle \chi_V, \chi_V \rangle = 1 + \langle \chi_W, \chi_W \rangle$ . On the other hand, if  $V' = \mathbb{k}(X \times X)$ , we see that a fixed point of  $g$  on  $X \times X$  is the same as a pair of fixed points of  $g$  on  $X$ , i.e.

$$\chi_{V'}(g) = \text{Fix}(g)^2 = \chi_V(g)^2.$$

It is clear that  $\Delta X$  is an isomorphic copy of  $X$  in  $X \times X$  and, since  $X$  is a transitive  $G$ -set, that this copy is in fact an orbit of the  $G$ -action. Using that  $\chi_V$  is real valued, we deduce that

$$\langle \chi_V, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_V(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)^2 = \langle \chi_{V'}, 1 \rangle$$

so that  $\langle \chi_V, \chi_V \rangle = 2$  if and only if the number of orbits of  $G$  on  $X \times X$  is 2, if and only if the action of  $G$  on  $X$  is doubly transitive. By the first item, we deduce that  $X$  is a doubly transitive  $G$ -set if and only if  $\langle \chi_W, \chi_W \rangle = 1$ , if and only if  $W$  is irreducible.

5. Determine the character tables of  $D_4$  and of  $Q_8$ . What do you observe?

5. *Solution.* Let us consider the presentation  $D_4 = C_4 \rtimes C_2$  where the generator of  $C_4$  is  $r$  and that of  $C_2$  is  $s$ . One can compute directly that  $D'_4 = \langle r^2 \rangle$  so  $D_4^{\text{ab}}$  is isomorphic to  $C_2 \times C_2$ . Similarly, one can compute directly that  $Q'_8 = \langle -1 \rangle$  and that  $Q_8^{\text{ab}} \cong C_2 \times C_2$ . Thus, all the one dimensional representations of both groups are equal to the one dimensional representations of  $C_2 \times C_2$ . Since both groups have the same cardinality, the next irreducible representation must have dimension at least two, and in fact at most two since  $8 - 4 = 4 = 2^2$ . Since the first four rows out of the five rows of their character tables are equal, it follows that the fifth row is also equal by the orthogonality relations. By looking at the sizes of the corresponding conjugacy classes, we see that both groups have the same character tables. However, these groups are not isomorphic, for example because all subgroups of  $Q_8$  are normal, but  $D_4$  admits non-normal subgroups, such as  $\langle s \rangle$ . For completeness, the character table for  $D_4$  is thus (for  $Q_8$  we replace  $r^2$  by  $-1$  and  $r, s, sr$  by  $i, j, k$ ):

|                    | Conjugacy classes |       |     |     |      |
|--------------------|-------------------|-------|-----|-----|------|
|                    | 1                 | $r^2$ | $r$ | $s$ | $sr$ |
|                    | 1                 | 1     | 2   | 2   | 2    |
| $\mathbb{C}_{+,+}$ | 1                 | 1     | 1   | 1   | 1    |
| $\mathbb{C}_{+,-}$ | 1                 | 1     | 1   | -1  | -1   |
| $\mathbb{C}_{-,+}$ | 1                 | 1     | -1  | 1   | -1   |
| $\mathbb{C}_{-,-}$ | 1                 | 1     | -1  | -1  | 1    |
| $\mathbb{C}^2$     | 2                 | -2    | 0   | 0   | 0    |

The unique 2-dimensional representation of  $D_4$  is given by the action on  $\mathbb{C}^2$  where

$$r \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

while the unique 2-dimensional representation of  $Q_8$  is afforded by

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$