Representation Theory — WiSe 2022

Semisimplicity, the radical and the density theorem

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1. Use the density theorem to show that the characters of simple modules of an algebra A are linearly independent over an algebraically closed field.

Note the hypothesis that the field is of characteristic zero was superfluous. In the case of groups rings, this also guarantees linear independence, since we can use the inner product we defined on class functions and orthogonality.

1. *Solution.* Let $S_1,\dots,S_n$ be non-isomorphic simple A-modules, and let us consider the semi-simple A-module $S=\bigoplus_{i=1}^n S_i$. Since the base field $\Bbbk$ is algebraically closed, the density theorem ensures that the sum of structure maps

$$\bigoplus_{i=1}^n \rho_i : A \longrightarrow \bigoplus_{i=1}^n \mathrm{End}_{\Bbbk}(S_i)$$

is onto. Suppose that there exist scalars $\lambda_1,\dots,\lambda_n\in\Bbbk$ such that $\lambda_1\chi_1+\cdots+\lambda_n\chi_n=0$. Equivalently, for all $a\in A$ we have that $\lambda_1\operatorname{Tr}(\rho_1(a))+\cdots+\lambda_n\operatorname{Tr}(\rho_n(a))=0$. Since the map above is surjective, we can pick $a\in A$ for which $\operatorname{Tr}(\rho_i(a))=\delta_{1i}$, which gives us that $\lambda_1=0$. Continuing in this way, we deduce that $\vec{\lambda}=0$, as we wanted.

2. When $\Bbbk$ is algebraically closed and A is semi-simple, show that characters form a basis of the space $\hom(V_A,\Bbbk)$ of class functions where $V_A=A/[A,A]$ is the (linear) quotient of A by the subspace of commutators.

2. *Solution.* We know from the previous exercise that the characters of simple modules are linearly independent, so it suffices we show that they span the space of class functions $\hom(V_A,\Bbbk)$. Notice that if $A=\mathrm{Mat}(n,\Bbbk)$, independently of the base field $\Bbbk$, we have that $A/[A,A]=\Bbbk$, as $[A,A]$ consists precisely of the subspace of traceless matrices, which has codimension 1. Thus, if A is semi-simple and admits r isomorphism classes of simple modules, in which case A is a sum of r matrices rings over $\Bbbk$ —since this field is algebraically closed— we deduce that $V_A=A/[A,A]$ has dimension r. Thus, we have r linearly independent elements in V_A which must then form a basis.

3. We say an algebra A is left Artinian if every descending chain of left ideals in A stabilizes. Show that the Jacobson radical of an Artinian ring is nilpotent.

3. *Solution.* Let J be the Jacobson radical of A. The descending chain of left ideals $J \supseteq J^2 \supseteq \cdots$ must stabilize, so there exists n for which $P = J^{n+1} = J^n$. Let us assume that this ideal is non-zero. In particular, there exist non-zero left ideals $I \subseteq P$ for which $PI \neq 0$. Since A is Artinian there exist a non-zero ideal I minimal with these properties. There must exist some $x \in I$ for which $Px \neq 0$ so $I = Ax$ is principal by minimality. Choose $z \in P$ for which $zx = x$. Then $(1-z)x = 0$, but $1-z$ is invertible, so $x = 0$ which is a contradiction.

4. An algebra A is called (von Neumann) *regular* if for every $a \in A$ there is some $b \in A$ such that $aba = a$.

(i) Show that if A is regular then $M_2(A)$ is regular.
(ii) Show that if A is regular and $e \in A$ is idempotent, then eAe is regular.
(iii) Deduce that any matrix algebra with coefficients in a regular algebra is regular.

4. *Solution.* First, notice that if x, y are elements of an algebra such that $xyx - x$ is a regular, then x is regular. Indeed, if z is such that $(xyx-x)z(xyx-x) = xyx-x$, then also

$$x(y - z - yxzxy + yxz + zxy)x = x.$$

Let α be any matrix in $M_2(A)$ and suppose that r witnesses that α_{21} is regular. Then one can check that $\alpha\beta\alpha - \alpha$ is upper-triangular where β is the strictly upper-triangular 2×2 matrix with only non-zero entry r. Thus, it suffices to show that upper-triangular matrices with coefficients in A are regular. For such a matrix α, suppose that s and t witness that α_{11} and α_{22} are regular. A similar computation with $\beta = \mathrm{diag}(s,t)$ shows that $\alpha\beta\alpha - \alpha$ is strictly upper-triangular. In this case, if $a \in A$ is the only non-zero entry of such a matrix α and β is the strictly *lower* triangular matrix whose non-zero coefficient witnesses that a is regular, we directly compute that $\alpha\beta\alpha = \alpha$.
For the second point, suppose that $x \in eAe$ and pick $y \in A$ such that $xyx = x$. Then $eye = y'$ is such that $xy'x = x$ since $xe = ex = x$, and eAe is regular. To conclude, consider $M_n(A)$ and pick m for which $n \leqslant k = 2^m$. Then the matrix e that consists of a block identity $n \times n$ matrix and zeros elsewhere is idempotent in $M_k(A)$ and $eM_k(A)e \cong M_n(A)$, so the algebra $M_n(A)$ is regular.

5. A $\Bbbk$-linear representation V of G is absolutely irreducible if for every field extension $K/\Bbbk$, the representation $V_K = K \otimes_{\Bbbk} V$ is irreducible. Suppose that $|G|$ is invertible in $\Bbbk$ and V is a $\Bbbk$-linear representation. Then V is irreducible if and only if $\mathrm{End}_G(V)$ is a division algebra, and it is absolutely irreducible if and only if $\mathrm{End}_G(V)$ consists of the multiples of the identity (i.e. it is isomorphic to $\Bbbk$).

5. *Solution.* If V is irreducible then we know its endomorphism algebra is a division algebra. If V is reducible, then by Maschke's theorem there is a projection π that is equivariant but not 1_V, so $\pi(1-\pi)=0$ and $\mathrm{End}_G(V)$ is not a division algebra.

Suppose that V is absolutely irreducible so that $D_K = \mathrm{End}_G(V_K)$ is a division algebra for every extension field K of $\Bbbk$. This implies that $D_{\Bbbk} = \Bbbk$. If not, pick some endomorphism $c \in D_{\Bbbk}$ that is not in $\Bbbk$ and consider its minimal polynomial f over $\Bbbk$, which must be of degree > 1. Since $D_{\Bbbk}$ is a division algebra f is irreducible —otherwise, we could decompose V into invariant subspaces— and $K = \Bbbk[x]/(f)$ is a field. But then D_K is not a division algebra since f is reducible in K, which cannot be. Conversely, if $D_{\Bbbk} = \Bbbk$ then $D_K = K$ is a division algebra and V_K is irreducible for every extension K of $\Bbbk$.