

Representation Theory — WiSe 2022

Restriction, induction and Frobenius reciprocity

1. Let $\mathcal{S} = \{s_1, \dots, s_r\}$ be a cross section of G with respect to H and let V be an H -module with linear basis $\mathcal{B}$. Define a map $\sigma : G \rightarrow S_r$ and a map $\mu : [r] \times G \rightarrow H$ by the formula $gs_i = s_{\sigma_g(i)}\mu_i(g)$.

- (i) Show that for each $g \in G$, the matrix of g with respect to the basis $\mathcal{S} \times \mathcal{B}$ of $\text{Ind}(V)$ is obtained by permuting the blocks of the block diagonal matrix $\text{diag}([\mu_1(g)]_{\mathcal{B}}, \dots, [\mu_r(g)]_{\mathcal{B}})$ according to σ_g .
- (ii) Show that block i is fixed by σ_g if and only if $s_i^{-1}gs_i \in H$, in which case the matrix appearing on this block is precisely that of $s_i^{-1}gs_i$ in base $\mathcal{B}$.
- (iii) Show that in this case $h^{-1}gh \in H$ for all h in the coset of s_i and thus obtain the formula for the character of an induced representation we gave in **Lecture 5**, namely

$$\chi_{\text{Ind}(V)}(g) = \frac{1}{|H|} \sum_{h \in H} \dot{\chi}_V(h^{-1}gh)$$

where the dot denotes the extension by zero to G .

1. *Solution.* We have proved in class that if $(v_1, \dots, v_n)$ is a basis of V then $\text{Ind}(V)$ has a basis $(s_i \otimes v_j : i \in [r], j \in [n])$ where we order $s_i \otimes v_j$ in the dictionary order. It follows from the way g acts on $\mathcal{S}$ that

$$g(s_i \otimes v_j) = s_{\sigma_g(i)} \otimes \mu_i(g)v_j.$$

This is precisely saying that up to the permutation given by σ_g , the action of g on $\text{Ind}(V)$ breaks down into the requisite blocks, which addresses (i). To prove (ii), notice that $gs_i = s_i\mu_i(g)$ if and only if $\sigma_g(i) = i$ in which case $s_i^{-1}gs_i = \mu_i(g) \in H$. Thus, the matrix appearing in this block is that of $s_i^{-1}gs_i$ in base $\mathcal{B}$. To conclude, we notice that if $h \in s_iH$ then $h = s_ih_1$ for some $h_1 \in H$ so $h^{-1}gh \in H$ since H is stable by conjugation by h_1 . Thus, taking traces for the action of g on $\text{Ind}(V)$ and noting that only the blocks fixed by σ_g contribute to the trace, we deduce that

$$\chi_{\text{Ind}(V)}(g) = \sum_{i=1}^r \chi_V(s_i^{-1}gs_i) = \frac{1}{|H|} \sum_{h \in H} \dot{\chi}_V(h^{-1}gh).$$

2. Let G be a finite group and let H be a subgroup. If V is a G -module, show that V is isomorphic to a module induced from an H -module U if and only if it contains an H -submodule isomorphic to U such that $V = g_1U \oplus \cdots \oplus g_rH$ for $\{g_1, \dots, g_r\}$ some cross section of G relative to H .

2. Solution. Let W be an H -module, and suppose that $\{g_1 = 1, \dots, g_n\}$ is a cross section of G relative to H . We know that $\mathbb{k}G$ is a free right $\mathbb{k}H$ -module with basis $\{g_1, \dots, g_n\}$, so that we have an isomorphism of $\mathbb{k}$ -vector spaces

$$\text{Ind}(W) = \mathbb{k}G \otimes_{\mathbb{k}H} W \cong g_1W \oplus \cdots \oplus g_nW$$

where for each $i \in [n]$ we write g_iW for the summand corresponding to $g_i \otimes W$, and for which the action is given by the rule $gg_i \otimes w = g_{\sigma_g(i)} \otimes \mu_i(g)w$. Without loss of generality, we can assume that $g_1 = 1$ is the representative for the coset H , in which case it is clear that if $h \in H$ and $w \in W$ then $h(1 \otimes w) = h \otimes w = 1 \otimes hw \in 1 \times W = W$. Thus, $\text{Ind}(W)$ admits an H -invariant copy of W giving the requisite direct sum decomposition. Conversely, suppose that $V = g_1W \oplus \cdots \oplus g_nW$ admits such a decomposition. We can again assume that $g_1 \in H$ and thus, since W is H -invariant, that $g_1 = 1$. On the summand g_iW , we compute that the action on g_iw is given by $g \cdot (g_iw) = g_{\sigma_g(i)}(\mu_i(g)w) \in g_{\sigma_g(i)}W$, and thus tracing back our steps we see that V is isomorphic with $\text{Ind}(V)$.

3. Let $g \in G$ be any element of finite order n , and consider the subgroup $H = \langle g \rangle$ defined by g . Let ω be an n -th root of unit, and consider the induced representation $V = \text{Ind}_H^G(\mathbb{C}_\omega)$ where H acts on $\mathbb{C}_\omega$ by $g\lambda = \omega\lambda$.

- (i) Show that $\chi_V(h) = \frac{1}{n} \sum_{i=0}^{n-1} c_g(h, i) \omega^i$ where for each $i \in \{0, \dots, n-1\}$ we set $c_g(h, i) = |\{x \in G : xhx^{-1} = g^i\}|$.
- (ii) Show that if g and g' are not conjugate, then $c_g(g', 1) = 0$ and $c_g(g, 1) \geq 1$, and conclude that there must exist an n th root of 1 for which $\chi_V(g) \neq \chi_V(g')$.
- (iii) Deduce that g and g' are conjugate in G if and only if they have the same value over all irreducible complex representations.

3. Solution. For the first item, let us consider the formula for the induced representation, and notice that for each $h \in G$ such that $xhx^{-1} \in H = \langle g \rangle$, say $xhx^{-1} = g^i$, we have that $\chi_{\mathbb{C}_\omega}(xhx^{-1}) = \omega^i$. Collecting terms according to the power i and the number of such $x \in G$, we see that $\chi_V(h)$ is precisely the polynomial in ω given by

$$P_{g,h}(\omega) = \frac{1}{n} \sum_{i=0}^{n-1} c_g(h, i) \omega^i.$$

For the second point, we consider the polynomials $P_{g,g}$ and $P_{g,g'}$, and notice that the difference $P_{g,g}(z) - P_{g,g'}(z)$ is a polynomial of degree at most n which is not the zero polynomial, since its

linear coefficient is $c_g(g, 1) - c_g(g', 1) = c_g(g, 1) \geq 1$. Thus, there must exist a root of unit ω so that if $V = \text{Ind}_H^G(\mathbb{C}_\omega)$, then

$$\chi_V(g) = P_{g,g}(\omega) \neq P_{g,g'}(\omega) = \chi_V(g').$$

To conclude, we know that characters are class functions so if g and g' are conjugate, then they take the same value on every character. Conversely, if g and g' are not conjugate, we see that there exists a character on which they differ and, since any character is a sum of irreducible characters, that in fact g and g' must take different values in at least one irreducible character of G .

4. Suppose that $G \rightarrow H$ is a surjection (instead of an injection into a subgroup) with kernel $N \triangleleft G$. Show that if V is a G -module, then V_N is an H -module isomorphic with $\text{Ind}(V)$. What is its character?

4. Solution. Let us show that $\text{Ind}(V)$ is given by $V_N = V/U$, the quotient of V by the subspace $U = \langle v - nv : v \in V, n \in N \rangle$. By construction, N acts trivially on the G -module V_N , and this is the smallest quotient of V on which N acts trivially. Notice that since N is normal in G , the subspace U is in fact G -invariant. To continue, notice that since N acts trivially on V_N , we have an induced action of the quotient group $G/N \cong H$ on it. To prove that V_N is isomorphic to $\text{Ind}(V)$, it is sufficient to show that for any H module W we have a natural isomorphism

$$\alpha_{V,W} : \text{hom}_G(V, \text{Res}(W)) \rightarrow \text{hom}_H(\text{Ind}(V), W).$$

But if $f : V \rightarrow W$ is G -linear, we see that f maps U to zero, since N acts trivially on W and thus $f(v - nv) = f(v) - nf(v) = 0$. It follows that f induces a map $\bar{f} : V_N \rightarrow W$ which is a map of H -modules. Conversely, by pre-composing with the quotient map $V \rightarrow V_N$, we obtain the inverse arrow.

Let us now assume that $|N|$ is invertible in $\mathbb{k}$, and consider the subspace of N -invariants

$$V^N = \{v \in V : nv = v \text{ for all } n \in N\}.$$

The composition $V^N \rightarrow V \rightarrow V_N$ is a map which is an isomorphism. Indeed, the inverse map $V_N \rightarrow V^N$ sends the class $\bar{v} \in V_N$ to the element $i(v) = \frac{1}{|N|} \sum_{n \in N} nv$. Thus, if $v_1, \dots, v_s$ is a linearly independent set in V that induces a basis for V_N , the elements $i(v_1), \dots, i(v_s)$ are a basis of V^N , and hence the character of V_N is given by the formula

$$\chi_{V_N}(g) = \frac{1}{|N|} \sum_{g' \in gN} \chi_V(g').$$

Add: if we identify H with the quotient G/N and the map $G \rightarrow H$ with the projection $\pi : G \rightarrow$

G/N , we can give a description of $\mathbb{k}[H]$ in terms of $\mathbb{k}[G]$, which gives a transparent proof that V_N is indeed $\text{Ind}(V)$.

Let us consider the left ideal I_N in $\mathbb{k}G$ generated by the elements of the form $1 - n$ where $n \in N$. Because N is normal, it is in fact a two sided ideal, since for any $g \in G$ and any $n \in N$ we have that $(1 - n)g = g - ng = g - gn' = g(1 - n') \in I_N$. The projection π induces a $\mathbb{k}$ -algebra map $\pi : \mathbb{k}G \rightarrow \mathbb{k}[G/N]$ and we claim that $\ker \pi = I_N$ so that we have an isomorphism of $\mathbb{k}$ -algebras between $\mathbb{k}[G/N]$ and $\mathbb{k}G/I_N$. It is clear that $\pi(I_N) = 0$ since $n \mapsto 1$ for each $n \in N$, so π at least induces a surjective map $q : \mathbb{k}G/I_N \rightarrow \mathbb{k}[G/N]$.

Notice now that $\{1\} \cup \{g - 1 : g \in G\}$ is a basis of $\mathbb{k}G$, and that if $g \in G$ and we choose a cross section $1 = g_1, \dots, g_s$ of G relative to N , then there exists a unique $i \in [s]$ such that $\pi(g) = \overline{g_i}$. Moreover, the set of classes $\{1\} \cup \{1 - \overline{g_i} : i \in [s]\}$ forms a basis of $\mathbb{k}[G/N]$ as well. Hence, if we pick a linear combination $\lambda + \sum_{g \in G \setminus 1} \lambda_g(1 - g)$ in the kernel of π , which we can write as

$$\lambda + \sum_{i=1}^s \sum_{n \in N \setminus 1} \lambda_{g_i n}(1 - g_i n)$$

we obtain first that $\lambda = 0$, and that on each coset C of N , $\sum_{g \in C} \lambda_g = 0$. Indeed, passing to the image of π we get

$$\lambda + \sum_{i=1}^s \sum_{n \in N \setminus 1} \lambda_{g_i n}(1 - \overline{g_i}) = 0.$$

To conclude, we see that we now can write

$$\sum_{i=1}^s \sum_{n \in N \setminus 1} \lambda_{g_i n}(1 - g_i n) = \sum_{i=1}^s \sum_{n \in N \setminus 1} \lambda_{g_i n} g_i(1 - n) \in I_N$$

which is what we wanted. Finally, we see that $\text{Ind}(V) = \mathbb{k}[G]/I_N \otimes_{\mathbb{k}[G]} V = V/I_N V$ and that $I_N V$ is precisely the G -submodule U we defined above, since I_N is a two sided ideal.

5. Compute the character table of A_5 by considering the index two subgroup H generated by all double transpositions and the 5-cycle (12345).

5. Solution. There are five conjugacy classes in A_5 , whose representatives are 1, (12)(34), (123), (12345) and $(12345)^2$. Notice that these result from four conjugacy classes in S_5 , one of which splits into two conjugacy classes in A_5 . We can begin by looking at the character χ_1 of the trivial representation and the regular representation χ on $\{1, 2, 3, 4, 5\}$.

	1	(12)(34)	(123)	(12345)	(13524)
	1	15	20	12	12
χ_1	1	1	1	1	1
χ	5	1	2	0	0

A direct computation, similar to what we have done before, shows that the regular representation decomposes as $\chi_1 + \chi_2$ where χ_2 is irreducible. To obtain the three missing characters, let us consider the subgroup H , and notice that $u = (12345)$ generates an index two (normal) subgroup in H .

Thus, we have two characters λ_1, λ_2 of H corresponding to one dimensional representations, the trivial representation and the “alternating” representation corresponding to the quotient map from H onto $H/\langle u \rangle \cong C_2$. In turn, we check directly that if $w = (13245)$ then

$$C = \{1, w, w^2, w^3, w^4, (12)(34)\}$$

is a cross-section of A_5 relative to H . We now use the formula for the induced character, and compute the following values for the induced characters

	1	(12)(34)	(123)	(12345)	(13524)
	1	15	20	12	12
λ'_1	6	2	0	1	1
λ'_2	6	-2	0	1	1

With this at hand, we can compute that λ'_1 and λ'_2 are orthogonal, so they have no common irreducible factors. Moreover, we also compute that

$$(\lambda'_1, \lambda'_1) = 2, \quad (\lambda'_2, \lambda'_2) = 2, \quad (\lambda'_1, \chi_1) = 1.$$

All of this means that $\lambda'_1 = \chi_1 + \chi_3$ where χ_3 is the character of an irreducible representation, and that $\lambda'_2 = \chi_4 + \chi_5$ where χ_4, χ_5 are the characters of the last two missing irreducible representations. Since we know the representations for χ_1, χ_2, χ_3 have dimension 1, 4 and 5, respectively, we deduce that

$$\chi_4(1)^2 + \chi_5(1)^2 = 18$$

which is only possible if both values are 3. Thus, so far, we have the following table:

	1	(12)(34)	(123)	(12345)	(13524)
	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	a	b	c	d
χ_5	3	a'	b'	c'	d'

To conclude, we notice that (12345) is conjugate in A_5 to its inverse, so any character takes real values on it. The orthogonality relations for χ_4 and the first three characters give us three linear equations on (a, b, c, d) plus a polynomial equation, and the fact that $\chi_4 + \chi_5 = \lambda'_2$ determines (a', b', c', d') in terms of (a, b, c, d) . Solving this system of equations completes the table, which is as follows, where φ, φ' are the two (real) solutions to $t^2 = t + 1$:

	1	(12)(34)	(123)	(12345)	(13524)
	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	φ	φ'
χ_5	3	-1	0	φ'	φ

For completeness, the system of linear equations is:

$$\begin{aligned}
15a + 20b + 12c + 12d &= -3 \\
15a - 20b &= -15 \\
+ 20b - 12c - 12d &= -12
\end{aligned}$$

One can check directly that they imply that $b = 0$ and that $a = -1$, and that the last equation is $c + d = 1$. The equation $\langle \chi_4, \chi_4 \rangle = 1$ gives us that c is a solution to $c^2 = c + 1$, in which case d is the conjugate root, like we wanted.