

Representation Theory — WiSe 2022

Representation theory of symmetric groups

1. Hook length formula. If λ is a partition and we pick a single box b in its diagram, the hook length h_b is given by the number of boxes directly below or directly to the right of b , counting b only once. We write h_λ for the product $\prod_{b \in \lambda} h_b$.

- (i) Show that $\dim V_\lambda$ is given by $\frac{n!}{\rho!} \prod_{1 \leq i < j \leq k} (\rho_i - \rho_j)$ where $\rho = \lambda + \delta$.
- (ii) Proceeding by induction on the number of columns of λ , show that $\dim V_\lambda = n! / h_\lambda$.

Hint: For the first item, use Frobenius' formula. For the second item, consider the first **row** R of λ and write $\prod_{b \in R} h_b$ explicitly. What do you observe?¹

1. By definition we have that $\dim V_\lambda = \chi_\lambda(C_\mu)$ where $\mu = (1^n)$ is the cycle type of the identity, so the Frobenius character formula tells us that this is equal to the coefficient of $x^{\lambda+\delta} = \prod_i x_i^{\lambda_i+k-i}$ in $a_\delta p_\mu$. Here a_δ is the Vandermonde determinant polynomial in k variables $x = (x_1, \dots, x_k)$ which we can write as

$$a_\delta(x) = \sum_{\sigma \in S_k} (-1)^\sigma \prod_{j=1}^k x_j^{k-\sigma(j)}$$

Since $p_\mu(x) = (x_1 + \dots + x_k)^n$ the multinomial theorem tells us that $p_\mu(x) = \sum \frac{n!}{i_1! i_2! \dots i_k!} x^{\mathbf{i}}$ so that if we multiply out $a_\delta p_\mu$ we get a sum of terms of the form

$$(-1)^\sigma \frac{n!}{i_1! i_2! \dots i_k!} x_1^{k-\sigma(1)+i_1} x_2^{k-\sigma(2)+i_2} \dots x_k^{k-\sigma(k)+i_k}.$$

For all $j \in \{1, \dots, k\}$ we have $k - \sigma(j) + i_j = \lambda_j + k - j$ if and only if $i_j = \lambda_j + \sigma(j) - j$, so the coefficient of $x^{\lambda+\delta}$ in $a_\delta p_\mu$ is equal to

$$\sum_{\sigma \in M_\lambda} (-1)^\sigma \frac{n!}{\prod_j (\lambda_j + \sigma(j) - j)!}$$

¹The proof below uses the first row instead of the first column, but since $\dim V_\lambda$ is conjugation invariant, one can rewrite it to use the first column, too!

where M_λ consists of those $\sigma \in S_k$ such that $\lambda_j + \sigma(j) - j \geq 0$ for all $j \in \{1, \dots, k\}$. Note that also that $\lambda_j + \sigma(j) - j \geq 0$ if and only if $\rho(j) - k + \sigma(j) \geq 0$. Moreover, for all $\sigma \in M$

$$\frac{\prod_j \rho_j \cdot (\rho_j - 1) \cdots (\rho_j - k + \sigma(j) + 1)}{\rho_1! \cdots \rho_k!} = \frac{1}{\prod_j (\rho_j - k + \sigma(j))!}.$$

and, since $\rho_j - k = \lambda_j - j$, we get this coefficient is equal to

$$\frac{n!}{\rho_1! \cdots \rho_k!} \sum_{\sigma \in S_k} (-1)^\sigma \prod_j \rho_j \cdot (\rho_j - 1) \cdots (\rho_j - k + \sigma(j) + 1).$$

Notice that we can take the sum over all of S_k , since the new terms do not contribute to the sum. Let $\tau \in S_k$ be the permutation (written in one line notation) $(k, k-1, \dots, 1)$ so that $\tau\sigma(j) = k - \sigma(j) + 1$. We see then that we can rewrite the coefficient instead as

$$\text{sgn}(\tau) \frac{n!}{\rho_1! \cdots \rho_k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_j \rho_j \cdot (\rho_j - 1) \cdots (\rho_j - \sigma(j) + 2)$$

where the sum is the determinant of the matrix $A(\rho)$ with general coefficient

$$A(\rho)_{ji} = \prod_{r=1}^{i-1} (\rho_j + 1 - i + r).$$

By adding the second column to the third, and so forth, we see that we can reduce this matrix through column operations to the usual Vandermonde determinant for the scalars $(\rho_1, \dots, \rho_k)$, that is, this coefficient is in fact

$$\text{sgn}(\tau) \frac{n!}{\rho_1! \cdots \rho_k!} \prod_{1 \leq i < j \leq k} (\rho_j - \rho_i) = \frac{n!}{\rho_1! \cdots \rho_k!} \prod_{1 \leq i < j \leq k} (\rho_i - \rho_j)$$

where the last equality comes from the fact the determinant changes one more time by $\text{sgn}(\tau)$ if we reverse the order of the variables. This proves the first item.

To prove the hook length formula using the " ρ -formula", we will use the following interesting result in MacDonal's *Symmetric Functions and Hall Polynomials* (page 3): if λ is any partition, $m \geq \lambda'_1$ and $n \geq \lambda_1$, then the $m+n$ numbers

$$\lambda'_i + n - i \quad (1 \leq i \leq n), \quad n - 1 + j - \lambda_j \quad (1 \leq j \leq m)$$

are a permutation of $\{0, 1, 2, 3, \dots, m+n-1\}$. To see how this implies the result, let us note that the hook lengths appearing along the first row of λ are precisely $\lambda_1 + \lambda'_i - i$ for $1 \leq i \leq \lambda_1$ while the numbers appearing in the product $\prod_{1 < j \leq k} (\rho_1 - \rho_j)$ are precisely $\lambda_1 + j - 1 - \lambda_j$ for $1 < j \leq k$, where 0 would appear if we allow $j = 1$. Taking $n = \lambda_1$ and $m = \lambda'_1 (= k)$ in the result means that

the product of these two integers is equal to $\rho_1! = (\lambda_1 + k - 1)!$, which is what we wanted to prove. Deleting the first row of the partition, we may now proceed by induction.

To prove the claim above, note that the diagram of λ' is contained inside the diagram of the partition (m^n) , which is an $n \times m$ rectangle. Number the successive segments of the boundary line between λ' and its complement in (m^n) with the numbers $0, 1, \dots, m + n - 1$, starting at the bottom left. We then note that numbers assigned to the vertical segments are $\lambda'_i + n - i$ and by transposition those assigned to the horizontal segments are $(m + n - 1) - (\lambda_j + m - j) = n - 1 + j - \lambda_j$ for $1 \leq j \leq m$.

2. Using the determinantal formula for the Schur functions given during the lectures, show that the following *Pieri formula* holds:

$$s_\mu e_r = \sum_{\lambda} s_\lambda$$

where the sum ranges through all partitions where $\lambda - \mu$ is a vertical r -strip. In turn:

- (i) Apply the involution ω to this equation to obtain a formula involving horizontal strips and the complete homogeneous symmetric functions instead.
- (ii) Deduce the decomposition of $\text{Ind}_{S_n}^{S_{n+1}} V_\lambda$ into irreducible components: the irreducible modules that appear are those corresponding to partitions that can be obtained by adding one box to λ .

2. Recall that $e_r = \sum_{\alpha} x^\alpha$ where α ranges through the set I_r of all $(0, 1)$ -vectors whose sum is r , so using the determinantal formula for the Schur functions we can begin to write

$$a_{\mu+\delta} e_r = \sum_{\sigma \in S_n} (-1)^\sigma x^{\sigma(\mu+\delta)} \sum_{\alpha \in I_r} x^\alpha = \sum_{\alpha \in I_r} a_{\mu+\alpha+\delta}.$$

Since $\mu + \delta$ is strictly decreasing, adding 0s or 1 in each coordinate can at worst make it non-increasing, so each term $\mu + \alpha + \delta$ is a partition, and we know that $a_{\mu+\alpha+\delta} = 0$ unless all terms in $\mu + \alpha + \delta$ are different or, what is the same, if $\mu + \alpha = \lambda$ is a honest partition. Thus $\lambda - \mu = \alpha$ is a vertical r -strip, which is precisely what we wanted to prove after dividing by a_δ throughout. We can apply ω to the equation above, which corresponds to conjugating the resulting partitions and interchanges e and h to obtain that

$$s_\mu h_r = \sum_{\lambda} s_\lambda$$

where now $\lambda - \mu$ is a horizontal r -strip. Finally, using the fact that the character of $\text{Ind}_{S_n}^{S_{n+1}} V_\lambda$ is precisely $e_1 s_\lambda$, we deduce the decomposition into irreducibles of this induced representation.

3. Let St_n denote the standard representation $V_{(n-1,1)}$ of S_n . Let S_{n-1} correspond to the subgroup of permutations in S_n fixing n .

- (i) Show that the restriction of $\bigwedge^s \text{St}_n$ to S_{n-1} is isomorphic to $\bigwedge^{s-1} \text{St}_{n-1} \oplus \bigwedge^s \text{St}_{n-1}$.
- (ii) Deduce that $\text{Res}(\bigwedge^s \text{St}_n) \cong V_{(n-s,1^{s-1})} \oplus V_{(n-s-1,1^s)}$.
- (iii) Use Frobenius reciprocity and Pieri's rule to prove that $\bigwedge^s \text{St}_n$ is irreducible and corresponds to the partition $(n-s, 1^s)$.

3. Recall that the standard representation St_n of S_n has a basis $v_i = e_1 - e_i$ where $i \in [2, n]$. Thus, a basis for $\bigwedge^s \text{St}_n$ is given by the products $v_{i_1} \wedge \cdots \wedge v_{i_s}$ where $2 \leq i_1 < \cdots < i_s \leq n$. Since v_{i_1} already contains e_1 , we see that this basis is given by

$$e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_s} - e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s}$$

where again $2 \leq i_1 < \cdots < i_s \leq n$. Let us restrict this representation to S_{n-1} . We then have a direct sum decomposition $\text{Res}(\bigwedge^s \text{St}_n) = W_0 \oplus W_1$ where W_0 is spanned by those basis vectors where $i_s \leq n-1$ while W_1 is spanned by those basis vectors where $i_s = n$. It follows that W_0 is isomorphic with $\bigwedge^s \text{St}_{n-1}$, while W_1 is isomorphic with $\bigwedge^{s-1} \text{St}_{n-1}$, and we have obtained that

$$\text{Res} \left(\bigwedge^s \text{St}_n \right) \cong \bigwedge^s \text{St}_{n-1} \oplus \bigwedge^{s-1} \text{St}_{n-1}.$$

By induction, it follows that the restriction of $\bigwedge^s \text{St}_n$ to S_{n-1} is isomorphic with the sum Specht modules $V_{(n-s,1^{s-1})}$ and $V_{(n-s-1,1^s)}$. On the other hand, Pieri's rule for the *restriction* of Specht modules tells us that the Young diagrams appearing in $\text{Res}_{S_{n-1}}^{S_n} V_\mu$ are precisely those obtained by looking at those partitions obtained by removing one square from μ , from where it follows that $\text{Res}_{S_{n-1}}^{S_n} V_{(n-s,1^s)}$ admits the same decomposition as $\bigwedge^s \text{St}_n$. It follows in fact that $(n-s, 1^s)$ is the only partition whose restriction is isomorphic with such direct sum. Since $\bigwedge^s \text{St}_n$ is irreducible (see for example *Proposition 3.12* of in Fulton–Harris), there is thus no other option but for $\bigwedge^s \text{St}_n$ to correspond to $(n-s, 1^s)$.

4. Show that $U_{(n-j,j)}$ decomposes as the direct sum of all $V_{(n-i,i)}$ where $i \in \{0, \dots, j\}$. In particular, note that the decomposition has no repeated irreducible components.

4. We have noted that $U_{(n-j,j)}$ is the module induced from the trivial representation of $S_{n-j} \times S_j$, and we know that the trivial representation 1_j has Frobenius character h_j , so the Frobenius character of $U_{(n-j,j)}$ is $\text{ch}(1_{n-j} \cdot 1_j) = h_{n-j} h_j$. Since h_{n-j} is equal to the Schur polynomial for the trivial partition $(n-j)$, we are left to compute $h_j s_{(n-j)}$. But we know that this is equal to $\sum_\mu s_\mu$ where μ is a partition containing $(n-j)$ such that $\mu \setminus (n-j)$ is a horizontal j -strip. The only possibility is that we add some number of blocks at the end of the top row, and leave some

others to the second row, leaving us with a partition of the form $(n-i, i)$ where $0 \leq i \leq j$. We have checked that $U_{(n-j, j)}$ and the direct sum of all $V_{(n-i, i)}$ where $i \in \{0, \dots, j\}$ have the same Frobenius character, so they must be isomorphic.