

Representation Theory — WiSe 2022

Simple representations, Schur's Lemma and characters

1. Let G be a finite group and let $G' = [G, G]$ be its derived subgroup. Let $G^{\text{ab}} = G/G'$ be the Abelianization of G .

- (i) If V is a one-dimensional representation of G , show there exists a morphism of groups $\chi_V : G \longrightarrow \mathbb{k}^\times$ such that $gv = \chi_V(g)v$ for all $g \in G$ and $v \in V$.
- (ii) Show that if V' is another one dimensional representation, V and V' are isomorphic if and only if $\chi_V = \chi_{V'}$.
- (iii) Conversely, show that if $\psi : G \longrightarrow \mathbb{k}^\times$ is a morphism then there is V a one dimensional representation with $\chi_V = \psi$.
- (iv) Prove that there are exactly $|\text{hom}(G^{\text{ab}}, \mathbb{k}^\times)|$ one dimensional representations.
- (v) For each $n \geq 2$, determine all one dimensional representations of S_n .

2. Let V be a representation given by $\rho : G \longrightarrow \text{GL}(V)$ and W be a one dimensional representation given by $\chi_W : G \longrightarrow \mathbb{k}^\times$.

- (i) Show that $V \otimes W$ is isomorphic to $\rho_W : V \longrightarrow \text{GL}(V)$ for which $\rho_W(g) = \chi_W(g)\rho(g)$.
- (ii) Show it is irreducible if and only if V is irreducible.
- (iii) When are V and $V \otimes W$ isomorphic?

3. Let V and W be $\mathbb{k}$ -linear G -representations and assume $|G|$ is invertible in $\mathbb{k}$. Show that

$$\dim_{\mathbb{k}} \text{hom}_G(V, W) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g^{-1}).$$

In particular, show that $\chi_{V^*} = \chi_V \circ \iota$ where ι is the inversion map of G .

4. Show that any irreducible complex representation of a finite abelian group is one dimensional. Consider rotations of the plane $\mathbb{R}^2$ and $\mathbb{Z}/3$ to obtain counterexamples in case the ground field is not algebraically closed. What irreducible representations result?

5. Let V be an S_3 -module and consider the cyclic subgroup $C_3 = \langle (123) \rangle \subseteq S_3$.

- (i) Show that V decomposes into a direct sum $V_0 \oplus V_1 \oplus V_2$ of eigenspaces for (123) with eigenvalues $1, \omega$ and ω^2 for ω a primitive 3rd root of unity.
- (ii) Show that if $v \in V$ is an eigenvector of eigenvalue ω^i for (123) and τ is a transposition, then $\tau(v)$ is an eigenvector for (123) of eigenvalue ω^{2i} .
- (iii) Show that if $v \in V_0$ then $\langle v, \tau(v) \rangle$ is either one dimensional or decomposes as the sum of the trivial and the sign representation.
- (iv) Show that if $v \in V_1$ or V_2 then $\langle v, \tau(v) \rangle$ is isomorphic to the standard representation.

6. Let $V = \mathbb{C}^n$ be the permutation representation of S_n afforded by $X = \{1, \dots, n\}$. Let V_1 be the span of $e_1 + \dots + e_n$ and let $V_2 \cong \mathbb{C}^{n-1}$ be its orthogonal complement.

- (i) Show that V decomposes as a direct sum $V = V_1 \oplus V_2$.
- (ii) Show that $\text{hom}_{S_n}(V, V)$ is two dimensional.
- (iii) Use this to show that V_2 is irreducible.

7. Let X be a G -set and form $V = \mathbb{k}X$ where $\mathbb{k}$ has zero characteristic.

- (i) Show that $\dim V^G = |X/G|$, the number of orbits of G on X .
- (ii) Show that V^G is one dimensional if and only if X is a transitive G -set.
- (iii) Compute χ_V and $\langle \chi_V, 1 \rangle$ using the inner product formula.
- (iv) Deduce Burnside's Lemma, which gives formula for the number of orbits of G on X in terms of fixed point sets.

8. Continuing from the previous question, assume that G acts transitively on X , so that $V = \mathbb{k} \oplus W$. Let V' be the representation afforded by the diagonal action of G on $X \times X$.

- (i) Show that $\langle \chi_V, \chi_V \rangle = 1 + \langle \chi_W, \chi_W \rangle$ and that $\chi_V^2 = \chi_{W'}$.
- (ii) Show that G has at least one orbit in $X \times X$ given by the diagonal $\Delta X = \{(x, x) : x \in X\}$.
- (iii) Show that $\langle \chi_V, \chi_V \rangle = \langle \chi_{V'}, 1 \rangle$ and conclude that G acts doubly transitively on X if and only if W is irreducible.

9. Suppose that the characteristic of $\mathbb{k}$ does not divide the order of G . Let V be a representation along with a subrepresentation W , and let $\psi_0 : V \rightarrow V$ be a projection of V onto W .

- (i) Show that $\psi(v) = |G|^{-1} \sum_{g \in G} g^{-1} \psi_0(gv)$ is G -equivariant and $\psi^2 = \psi$.
- (ii) Deduce that in this case subrepresentations admit complements by showing W is complemented by $\ker \psi$.

10. Let $\mathbb{k} = \mathbb{F}_p$ and let G be a group whose order is divisible by p . Let $\varepsilon : \mathbb{k}G \longrightarrow \mathbb{k}$ be the unique $\mathbb{k}$ -linear map sending every $g \in G$ to $1 \in \mathbb{k}$, and $V = \ker \varepsilon$. Suppose that W is a subrepresentation of $\mathbb{k}G$, and let $w \in W$ be non-zero.

- (i) Show that V is a proper subrepresentation. What is its dimension?
- (ii) Show that $\sum_{g \in G} gw = \varepsilon(w) \sum_{g \in G} e_g$. Call this element w' , notice $w' \neq 0$ unless $w \in V$.
- (iii) Show that $w' \in W \cap V$, so that V admits no complement in $\mathbb{k}G$.

11. Over $\mathbb{k} = \mathbb{F}_2$, consider the representation of $\mathbb{Z}/2$ over $\mathbb{k}^2$ defined by sending the only non-trivial element to the endomorphism $T(u, v) = (u, u + v)$. Show that this representation is indecomposable but admits a non-trivial invariant subspace.

12. Let $\mathbb{k}$ be algebraically closed and V be a simple representation of G . Let $\psi : V \longrightarrow V$ be a G -equivariant endomorphism. Using Schur's Lemma, show there exists a scalar $\lambda \in \mathbb{k}$ such that $T = \lambda 1_V$. Conclude that in this case the division algebra $\text{End}_G(V)$ is isomorphic to $\mathbb{k}$.

13. Let V be a representation of G , and consider the tensor product $V \otimes V$.

- (i) Show that the natural action of S_2 on $V \otimes V$ commutes with the G -action.
- (ii) Show that the subspace of S_2 -invariant tensors $S^2(V)$ is a subrepresentation.
- (iii) Show it is G -complemented by the subspace $\Lambda^2(V)$.
- (iv) Compute their characters in terms of χ_V .

14. Describe all one-dimensional representations of the dihedral group on 4 vertices D_4 , the alternating group A_4 and the quaternion group Q_8 .

15. Let V denote the standard representation of S_3 . Determine the decomposition of $V^{\otimes n}$ for each $n \geq 2$ using characters.

16. Determine the character tables of D_4 and of Q_8 . What do you observe?

Note: selected exercises are **1, 5, 7, 8, 16**.