

Representation Theory — WiSe 2022

Semisimplicity, the radical and the density theorem

1. Use the density theorem to show that the characters of simple modules of an algebra A are linearly independent over an algebraically closed field of characteristic zero.
2. When $\mathbb{k}$ is algebraically closed and A is semi-simple, show that characters form a basis of the space $\text{hom}(V_A, \mathbb{k})$ of class functions where $V_A = A/[A, A]$ is the (linear) quotient of A by the subspace of commutators.
3. Let M be a sum of simple modules $\{S_i : i \in I\}$ and suppose that N is a submodule. Show that there exists a subset I' of I such that $M = N \oplus N'$ where $N' = \bigoplus_{i' \in I'} S_{i'}$.
4. Show any quotient and any submodule of a semisimple module is also semisimple.
5. Let M be an A -module. Show the following statements are equivalent:
 - (i) M is a direct sum of simple modules.
 - (ii) M is a sum of simple modules.
 - (iii) Every submodule of M admits a complement.
6. Show an A -module M is simple if and only if M is generated by any non-zero element $x \in M$, if and only if M is isomorphic to A/J where J is a maximal (left) ideal in A .
7. We say an algebra A is left Artinian if every descending chain of left ideals in A stabilizes. Show that the Jacobson radical of an Artinian ring is nilpotent.
8. Let A be a finite dimensional algebra. Show that the Jacobson radical is the largest nilpotent two-sided ideal of A .
9. An algebra A is called (von Neumann) *regular* if for every $a \in A$ there is some $b \in A$ such that $aba = a$.
 - (i) Show that if A is regular then $M_2(A)$ is regular.
 - (ii) Show that if A is regular and $e \in A$ is idempotent, then eAe is regular.
 - (iii) Deduce that any matrix algebra with coefficients in a regular algebra is regular.

10. Show that A is regular if and only if every ideal is generated by an idempotent. Deduce that a finite dimensional regular algebra is semi-simple. Show that division algebras are regular and deduce that finite dimensional semi-simple algebras are regular.

11. Let V be an irreducible complex representation of G .

- (i) Show that every $z \in Z(G)$ acts by a scalar on V .
- (ii) Show that $|\chi_V(z)| = \dim(V)$ if $z \in Z(G)$ and deduce that $\dim(V)^2 \leq [G : Z(G)]$.
- (iii) Show that if V is faithful, then $Z(G)$ is cyclic.

12. Suppose that L/K is a field extension and V is an L -linear representation of G . Show that if V is defined over K , then χ_V in fact takes values in K . Show that the converse statement is not true.

13. Show that the map $(V, W) \mapsto V \otimes W$ is a bijection between the pairs of complex irreducible representations of G_1 and G_2 and the complex irreducible representations of $G_1 \times G_2$.

14. A $\mathbb{k}$ -linear representation V of G is absolutely irreducible if for every field extension $K/\mathbb{k}$, the representation $K \otimes_{\mathbb{k}} V$ is irreducible. Suppose that $|G|$ is invertible in $\mathbb{k}$ and V is a $\mathbb{k}$ -linear representation. Then V is irreducible if and only if $\text{End}_G(V)$ is a division algebra, and it is absolutely irreducible if and only if $\text{End}_G(V)$ consists of the multiples of the identity (i.e. it is isomorphic to $\mathbb{k}$).

15. We say that $\mathbb{k}$ is a *splitting field* for G if every irreducible $\mathbb{k}$ -representation is absolutely irreducible. Show that $\mathbb{k}$ is a splitting field for G if and only if $\mathbb{k}G$ is isomorphic to a product of matrix algebras with coefficients in $\mathbb{k}$.

16. Let G be a group and $\mathbb{k}$ be a splitting field for G . For each irreducible character χ of G , consider the element

$$e_\chi = \sum_{g \in G} \chi(g)g^{-1}.$$

- (i) Show that these are complete orthogonal idempotents.
- (ii) Show e_{χ_V} gives the projection onto the V -isotypical component of any module.
- (iii) Explain how these orthogonal idempotents provide the requisite isomorphism of $\mathbb{k}G$ with a product of matrix algebras over $\mathbb{k}$.

Note: selected exercises are **1, 2, 8, 9, 14**.