

Representation Theory — WiSe 2022

Restriction, induction and Frobenius reciprocity

1. Let H be an Abelian subgroup of G . Show every complex representation of G has dimension at most $[G : H]$. Deduce that all complex irreducible representations of the dihedral group have dimension at most two.
2. Let G be the dihedral group of order $2n$, and consider the subgroup H generated by the rotation $r \in G$. Show that if V is a one dimensional complex irreducible representation of H , then $G \otimes_H V$ is an irreducible representation of G . Determine when these are isomorphic and classify all irreducible representations of G .
3. For any subgroup H of a group G , any H -module V and any G -module U show there is a ‘push-pull formula’ isomorphism $\text{Ind}(V) \otimes U \longrightarrow \text{Ind}(V \otimes \text{Res}(U))$. Show also that dualization and induction commute.
4. Let $\mathcal{S} = \{s_1, \dots, s_r\}$ be a cross section of G with respect to H and let V be an H -module with linear basis $\mathcal{B}$. Define a map $\sigma : G \longrightarrow S_r$ and a map $\mu : [r] \times G \longrightarrow H$ by the formula $gs_i = s_{\sigma_g(i)}\mu_i(g)$.
 - (i) Show that for each $g \in G$, the matrix of g with respect to the basis $\mathcal{S} \times \mathcal{B}$ of $\text{Ind}(V)$ is obtained by permuting the blocks of the block diagonal matrix $\text{diag}([\mu_1(g)]_{\mathcal{B}}, \dots, [\mu_r(g)]_{\mathcal{B}})$ according to σ_g .
 - (ii) Show that block i is fixed by σ_g if and only if $s_i^{-1}gs_i \in H$, in which case the matrix appearing on this block is precisely that of $s_i^{-1}gs_i$ in base $\mathcal{B}$.
 - (iii) Show that in this case $h^{-1}gh \in H$ for all h in the coset of s_i and thus obtain the formula for the character of an induced representation we gave in **Lecture 5**, namely

$$\chi_{\text{Ind}(V)}(g) = \frac{1}{|H|} \sum_{h \in H} \dot{\chi}_V(h^{-1}gh)$$

where the dot denotes the extension by zero to G .

5. Show that for any irreducible representation V and every subgroup H of G , there exists an irreducible representation U of H such that $V \subseteq G \otimes_H U$.

6. Let G be a finite group and let H be a subgroup. If V is a G -module, show that V is isomorphic to a module induced from an H -module U if and only if it contains an H -submodule isomorphic to U such that $V = g_1 U \oplus \cdots \oplus g_r H$ for $\{g_1, \dots, g_r\}$ some transversal of H in G .

7. Prove the following extension of Maschke's Theorem. Suppose that V is a G -module where G is not necessarily finite, but such that it contains a normal subgroup of finite index H for which V is H -semi-simple. Show that if $[G : H]$ is invertible in $\mathbb{k}$, then V is G -semisimple.

8. Let V be a G -module and W another G -module. Show that $W^* \otimes V$ admits a subrepresentation isomorphic to the trivial representation if and only if V admits a subrepresentation isomorphic to W .

9. *Double Coset Lemma.* Let H and K be subgroups of G . For each $g \in G$, consider the subgroup $H_g = gHg^{-1} \cap K$ of K . Show that the double coset KgH is a disjoint union of left cosets of the form $k_1 g H, \dots, k_q g H$ if and only if $\{k_1, \dots, k_q\}$ is a cross-section of K for H_g .

10. *Proof of Mackey's Decomposition Theorem.* Let Δ be the set of double K - H cosets in G . Let U be an U -module. For each $D \in \Delta$, consider the subspace $DU = \sum_{g \in D} gU$ of $\text{Ind}(U)$ where we identify U with $1 \otimes U$.

- (i) Show that $\text{Ind}_H^G(U) = \bigoplus_{D \in \Delta} DU$ and each DU is K -invariant.
- (ii) For $g \in D$, show that DU contains gU as an H_g -submodule.
- (iii) Prove that DU is obtained from gU by inducing from K to H_g .
- (iv) Obtain the Mackey Decomposition

$$\text{Res}_K^G \text{Ind}_H^G(U) \cong \bigoplus_{s \in \mathcal{D}} \text{Ind}_{H_s}^K(W_s)$$

where W_s is the 'twisted' version of W where shs^{-1} acts by h and $\mathcal{D}$ is a set of representatives for double cosets.

Hint: For (i), group double cosets into cosets and thus partition a cross section of G with respect to H appropriately. For (iii), use the "Double Cosets Lemma" to assume that the elements in the cross section corresponding to D are of the form $k_1 g, \dots, k_q g$.

11. *Another result of Mackey.* Let A and B be subgroups of G with respective representations U and W , and let $\mathcal{R}$ be a set of representatives for the double A - B cosets of G . Let $C_g = A \cap {}^g B$. Then

$$\text{Ind}_A^G(U) \otimes \text{Ind}_B^G(W) \cong \bigoplus_{g \in \mathcal{R}} \text{Ind}(\text{Res}_{C_g}^A(U) \otimes \text{Res}_{C_g}^{gB}(gW))_{C_g}^G.$$

Hint: use the push-pull formula and Mackey's Decomposition Theorem. Notice we are implicitly using the 'twisted' submodule gW discovered in the proof of the Decomposition Theorem.

12. Mackey's Criterion. Let $\mathbb{k}$ be algebraically closed and $\mathbb{k}G$ semisimple. Let V be an H -module for some subgroup of G . For each $g \notin H$, consider the subgroup $H_g = H \cap {}^g H$ of H , and write V^g for the restriction $\text{Res}(V)_{H_g}^H$ and V_g for the 'twisted' H_g -module obtained from V . Then $\text{Ind}(V)$ is irreducible if and only if

- (i) V is irreducible.
- (ii) For all $g \notin H$, we have that V^g and V_g are orthogonal.

Hint: Use Frobenius' Reciprocity and Mackey's Decomposition Theorem to explicitly compute the inner product $\langle \chi_{\text{Ind}(V)}, \chi_{\text{Ind}(V)} \rangle$. We say two representations are orthogonal if they share no simples in their decompositions, i.e. if their characters are orthogonal.

13. Let $g \in G$ be any element of finite order n , and consider the subgroup $H = \langle g \rangle$ defined by g . Let ω be an n -th root of unit, and consider the induced representation $V = \text{Ind}_H^G(\mathbb{C}_\omega)$ where H acts on $\mathbb{C}_\omega$ by $g\lambda = \omega\lambda$.

- (i) Show that $\chi_V(h) = \frac{1}{n} \sum_{i=0}^{n-1} c_g(h, i) \omega^i$ where for each $i \in \{0, \dots, n-1\}$ we set $c_g(h, i) = |\{x \in G : xhx^{-1} = g^i\}|$.
- (ii) Show that if g and g' are not conjugate, then $c_g(g', 1) = 0$ and $c_g(g, 1) \geq 1$, and conclude that there must exist an n th root of 1 for which $\chi_V(g) \neq \chi_V(g')$.
- (iii) Deduce that g and g' are conjugate in G if and only if they have the same value over all irreducible complex representations.

Note: this explicit construction is due to David E. Speyer. A non-explicit argument uses the fact characters form a basis for the space of class functions, and the class function which is the indicator of the conjugacy class of g is then a linear combination of irreducible representations.

14. Suppose that $G \rightarrow H$ is a surjection (instead of an injection into a subgroup) with kernel $N \triangleleft G$. Show that if V is a G -module and $|N|$ is invertible in $\mathbb{k}$, then the space of N -invariants V^N is an H -module isomorphic with $\text{Ind}_G^H(V)$. What is its character?

Hint: first show that the space of N -coinvariants V_N , the quotient of V by the subspace of vectors of the form $v - nv$ with $n \in N$ is always isomorphic to $\text{Ind}(V)$. Then, show V^N and V_N are isomorphic when $|N|$ is invertible in $\mathbb{k}$ and use this to get a formula for the induced character.

15. Compute the character table of A_5 by considering the index two subgroup H generated by all double transpositions and the 5-cycle (12345).

Note: selected exercises are **4, 6, 13, 14, 15**.