

Representation Theory — WiSe 2022

Representation theory of symmetric groups

1. Hook length formula. If λ is a partition and we pick a single box b in its diagram, the hook length h_b is given by the number of boxes directly below or directly to the right of b , counting b only once. We write h_λ for the product $\prod_{b \in \lambda} h_b$.

- (i) Show that $\dim V_\lambda$ is given by $\frac{n!}{\rho!} \prod_{1 \leq i < j \leq k} (\rho_i - \rho_j)$ where $\rho = \lambda + \delta$.
- (ii) Proceeding by induction on the number of columns of λ , show that $\dim V_\lambda = n! / h_\lambda$.

Hint: For the first item, use Frobenius' formula. For the second item, consider the first column C of λ and write $\prod_{b \in C} h_b$ explicitly in terms of $\lambda + \delta$. What do you observe?

2. Use the hook length formula to show that the only irreducible representations of S_n of dimension less than n are those corresponding to the partitions

$$(n), \quad (1^n), \quad (n-1, 1) \quad \text{or} \quad (2, 1^{n-2})$$

or to the small exceptions $(2, 2)$, $(2, 2, 2)$ and $(3, 3)$.

3. Show that the (irreducible) standard representation V of S_n (of dimension n) corresponds to the partition $(n-1, 1) \vdash n$. *Suggestion:* notice that the only partition that is larger than $(n-1, 1)$ in the natural order is (n) , and begin studying $U_{(n-1, 1)}$ instead.

If μ and λ are partitions and $\lambda \supset \mu$, meaning that the diagram of λ contains the diagram of μ , the “difference” $\lambda - \mu$ obtained by erasing the boxes of μ appearing in λ , is called a *skew diagram*. We say a skew diagram is a horizontal r -strip if it has exactly r boxes and at most one box in each column. A vertical r -strip is defined analogously.

4. Using the determinantal formula for the Schur functions given during the lectures, show that the following *Pieri formula* holds:

$$s_\mu e_r = \sum_{\lambda} s_\lambda$$

where the sum ranges through all partitions where $\lambda - \mu$ is a vertical r -strip. In turn:

- (i) Apply the involution ω to this equation to obtain a formula involving horizontal strips and the complete homogeneous symmetric functions instead.
 - (ii) Deduce the decomposition of $\text{Ind}_{S_n}^{S_{n+1}} V_\lambda$ into irreducible components: the irreducible modules that appear are those corresponding to partitions that can be obtained by adding one box to λ .
- 5.** Read about the Robinson–Schensted–Knuth algorithm and how it can be used to prove orthogonality of the Schur functions.
- 6.** Let St_n denote the standard representation $V_{(n-1,1)}$ of S_n . Let S_{n-1} correspond to those permutations in S_n fixing n .
- (i) Show that the restriction of $\bigwedge^s \text{St}_n$ to S_{n-1} is isomorphic to $\bigwedge^{s-1} \text{St}_{n-1} \oplus \bigwedge^s \text{St}_n$.
 - (ii) Deduce that $\text{Res}(\bigwedge^s \text{St}_n) \cong V_{(n-s, 1^{s-1})} \oplus V_{(n-s-1, 1^s)}$.
 - (iii) Use Frobenius reciprocity and Pieri’s rule to prove that $\bigwedge^s \text{St}_n$ is irreducible and corresponds to the partition $(n-s, 1^s)$.
- 7.** Consider the action of S_n on k -subsets of $[n] = \{1, \dots, n\}$ and the corresponding representation W_k that arises in this way.
- (i) Show that W_k and W_{n-k} are isomorphic.
 - (ii) Compute the dimension of $\text{Hom}_{S_n}(W_k, W_l)$ for all $k, l \in [n]$.
 - (iii) Show that $\text{End}_{S_n}(W_k)$ is a commutative algebra.
- 8.** Consider the action of S_4 on the vertices, edges or faces of the tetrahedron and of the cube (for a total of six permutation representations). Compute the characters of these representations, find multiplicities of irreducible representations in them, and describe the corresponding invariant subspaces where they are realised.
- 9.** Let $\mu = (1, \dots, 1) \vdash n$ be the partition corresponding to the sign representation of S_n and let λ' denote the conjugate partition of $\lambda \vdash n$. Use symmetrisers to prove that $V_{\lambda'} \cong V_\lambda \otimes V_\mu$.
- 10.** Show that $U_{(n-j,j)}$ decomposes as the direct sum of all $V_{(n-i,i)}$ where $i \in \{0, \dots, j\}$. In particular, note that the decomposition has no repeated irreducible components.
- 11.** Show that restriction from S_{m+n} to any subgroup $S_m \times S_n$ defines a diagonal $\Delta : R \longrightarrow R \otimes R$. Show it is compatible with the exterior product in the sense that $\Delta(fg) = \Delta(f)\Delta(g)$ and that it is associative. Compute Δ on the h , e and p bases.

Note: selected exercises are **1, 4, 6, 10, 11**.