Representation Theory — WiSe 2022

Complement for Tutorial 7

During the tutorial we focused on the computation of Auslander–Reiten translates of non-projective indecomposable modules over a bound quiver path algebra A. To do it, we used the following useful result:

Lemma 1. *Suppose that $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$ is a minimal projective presentation of an A-module M, where A is a finite dimensional algebra. Then, there exists a four term exact sequence*

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \longrightarrow \nu P_0 \longrightarrow \nu M \longrightarrow 0$$

In particular, we have that $\underline{\dim}\,\tau M = \underline{\dim}\,\nu P_1 - \underline{\dim}\,\nu P_0 + \underline{\dim}\,\nu M$.

Recall that $\nu(-) = D(-)^t$ is the *Nakayama functor*, and that if $P(a)$ is the indecomposable projective module corresponding to a vertex $a \in Q_0$ then $\nu P(a) \cong I(a)$ canonically. We are also using $\underline{\dim}$ to denote the *dimension vector* of a module, that is, the vector $\underline{\dim} M = (\dim M_a)_{a \in Q_0}$.

Proof. By definition, $\tau = D\,\mathrm{Tr}$ and we have a four term exact sequence

$$0 \longrightarrow M^t \longrightarrow P_0^t \longrightarrow P_1^t \longrightarrow \mathrm{Tr}\,M \longrightarrow 0.$$

Applying D to this gives the requisite exact sequence. Since the sequence is exact, applying the rank-nullity theorem several times gives us the equality of dimensions. □

The Cartan matrix. If A is a finite dimensional algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents, we define its *Cartan matrix* C_A to be the $n \times n$ integer matrix with $(C_A)_{ij} = \dim e_i A e_j$. Since indecomposable decompositions are unique up to permutations, C_A is unique up permutation.

Lemma 2. *If $A = \Bbbk Q/I$ is a bound quiver algebra and we enumerate its vertices with corresponding idempotents e_i for $i \in [n]$, then the ith column C_A is $\underline{\dim} P(i)$, the ith row is $\underline{\dim} I(i)$. Moreover, $C_A \underline{\dim} S(i) = P(i)$ and $C_A^t \underline{\dim} S(i) = \underline{\dim} I(i)$.*

The following gives a very interesting connection between resolutions of the simple modules and the invertibility of the Cartan matrix over the integers.

Proposition 1. *With A as before, suppose that each simple module $S(j)$ admits a finite projective resolution. That is, for each j there exists an exact sequence*

$$0 \longrightarrow P_{s,j} \longrightarrow P_{s-1,j} \longrightarrow \cdots \longrightarrow P_{1,j} \longrightarrow P_{0,j} \longrightarrow S(j) \longrightarrow 0$$

where all the $P_{i,j}$ are projective. Then the Cartan matrix C_A is invertible over $\mathbb{Z}$.

Proof. Taking dimensions in the sequence above, the rank-nullity theorem tells us that

$$\underline{\dim} S(j) = \underline{\dim} P_{0,j} - \underline{\dim} P_{1,j} + \cdots + (-1)^s \underline{\dim} P_{s,j}$$

Since each $P_{i,j}$ is a sum of indecomposable projectives, their dimension vectors are integer linear combinations of the columns of the Cartan matrix. Hence, we deduce that there exist a matrix B of integers b_{ij} such that

$$\underline{\dim} S(j) = \sum_{i=1}^{n} b_{ji} \underline{\dim} P(i).$$

This is precisely telling us that C_A admits B as an inverse, since the $S(j)$ are simply the coordinate vectors of $\mathbb{Z}^n$. □

When the Cartan matrix of A is invertible, we define the *Coxeter matrix of A* to be $\Phi_A = -C_A^t C_A^{-1}$. The following lemma allows us to get another formula for the dimension vector of τM. As recorded below in the exercises, whenever the Coxeter matrix of A is defined, we have that $\Phi_A(\underline{\dim} P) = -\underline{\dim} \nu P$ for every projective A-module.

Lemma 3. *Suppose that C_A is invertible. With the notation of the first lemma, we have that*

$$\underline{\dim} \tau M = \Phi_A(\underline{\dim} M) - \Phi_A(\underline{\dim} \ker p_1) + \underline{\dim} \nu M.$$

Proof. By the rank-nullity theorem, we have that $\underline{\dim} M - \underline{\dim} \ker p_1 = \dim P_0 - \underline{\dim} P_1$, so applying Φ_A and the exercise above, we see that $\Phi_A(\underline{\dim} M) - \Phi_A(\underline{\dim} \ker p_1) = \underline{\dim} \nu P_1 - \underline{\dim} \nu P_0$. If we compare this with the formula that we deduced in **Lemma 1**, we obtain the result. □

Hereditary algebras. We say that an algebra A is *hereditary* if every submodule of a projective module is projective.

Lemma 4. *A finite dimensional algebra is hereditary if and only if the radical of any (finite dimensional) projective module is projective.*

Proof. It suffices to show that a finite dimensional algebra A is hereditary if the radical of any projective module is projective. Let M be a submodule of a projective module P and let us proceed by induction on $\dim P$. If $\dim P = 1$, there is nothing to prove, so let us assume $\dim P > 1$ and the statement true for all projective modules of dimension less that $\dim P$. We can write $P = P_1 \oplus P_2$

where P_1 is indecomposable and P_2 may be zero. If $\pi : P \longrightarrow P_1$ is the projection and if $\pi(M) = P_1$, then the map $M \longrightarrow P_1$ is a split epimorphism (since P_1 is projective) and so $M \cong P_1 \oplus M'$ where $M' \cong M \cap P_2 \subseteq P_2$ is projective by induction. Thus, M is also projective. If not, then $M \subseteq \operatorname{rad} P_1 \oplus P_2$ is projective by induction since the $\operatorname{rad} P_1 \oplus P_2$ is projective by hypothesis and of dimension strictly less than P. □

In particular, if A is hereditary and finite dimensional and we choose a module M and a projective cover $p_0 : P_0 \longrightarrow M$, we see that $\ker p_0 = P_1$ is projective, so the minimal resolution of M has p_1 injective, This means in particular that A falls under the hypotheses of **Proposition 1**, and Φ_A is defined. We deduce that for any A-module

$$\underline{\dim}\, \tau M = \Phi_A(\underline{\dim} M) + \underline{\dim}\, \nu M.$$

In fact, we can say more:

Lemma 5. *Suppose that A is hereditary and M is indecomposable non-projective. Then νM vanishes and, consequently $\underline{\dim}\, \tau M = \Phi_A(\underline{\dim} M)$ for any non-projective module with no projective summand.*

Proof. It suffices that we show $M' = \hom_A(M, A) = 0$. Let us show that if M is indecomposable then every map $f : M \longrightarrow P$ with P projective is either zero of injective, this implies that every map from M to a projective is zero if M is non-projective indecomposable and A is hereditary, and in particular that every map $M \longrightarrow A$ is zero. If f is non-zero we have a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \operatorname{im} f \longrightarrow 0$$

with $\ker f \subsetneq M$. Since $\operatorname{im} f \subseteq P$ is projective, the sequence splits, and hence $M \cong \ker f \oplus \operatorname{im} f$. But M is indecomposable and $f \neq 0$, which means that $\ker f$ must be zero, so f is injective. □

Theorem 1. *Suppose that $A = \Bbbk Q$ is a quiver algebra of an ayclic quiver with no relations. The minimal projective resolution of the simple A-module $S(a)$ is of the form*

$$0 \longrightarrow \bigoplus_{\alpha : a \to b} \alpha \otimes P(b) \xrightarrow{\iota} P(a) \xrightarrow{\pi} S(a) \longrightarrow 0.$$

where the arrow is given by $\alpha \otimes \rho \longmapsto \alpha\rho$. Thus $\operatorname{rad} P(a) \cong \bigoplus_{\alpha : a \to b} \alpha \otimes P(b)$ and A is hereditary.

Proof. Let us compute the dimension of $\operatorname{rad} P(a)_c$ for each $c \in Q_0$. If $c = a$, then $\operatorname{rad} P(a)_c = 0$, while if $c \neq a$, then we know that $\operatorname{rad} P(a)_c = P(a)_c$. Since $\Bbbk Q$ has no relations, this vector space has dimension equal to the number of paths starting at a and ending at c. Any such path ρ can be written uniquely in the form $\rho = \alpha\rho'$ where $\alpha : a \to b$ is an arrow outgoing from a and ρ' is a

path $b \to c$. This is precisely the dimension of

$$\bigoplus_{\alpha : a \to b} \alpha \otimes P(b)$$

and i is surjective onto $\operatorname{rad} P(a)$, so it must be injective and the sequence is exact. □

Some exercises. The following exercises are intended to help you get more familiar with the computation of translates and the Coxeter matrix.

1. Use the projective presentation above to express the integer matrix Φ_A in terms of the paths counting function $w : Q_0 \times Q_0 \longrightarrow \mathbb{N}$ in case A is the quiver algebra of an acyclic quiver.

2. Let Q be the Kronecker quiver with two vertices 1 and 2 and two parallel arrows from 2 to 1.

(i) By counting arrows, or otherwise, show that $\Phi_A = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$.

(ii) Show that $\Phi_A^n = \begin{bmatrix} -2n+1 & 2n \\ -2n & 2n+1 \end{bmatrix}$.

(iii) Conclude that $\{\tau^i S(2) : i \geqslant 0\}$ is a family of pairwise non-isomorphic indecomposable modules with unbounded dimensions.

Thus, $\Bbbk Q$ is an algebra of *infinite* representation type: the number of isomorphism classes of finite dimensional indecomposable modules is infinite.

3. With Q as before, let M be the module with $M_1 = M_2 = \Bbbk$, and where the maps given by the zero map and the identity.

(i) Show that $\operatorname{End} M = \Bbbk$, so M is indecomposable (and not simple).
(ii) Show that there is an exact sequence $0 \longrightarrow \tau M \longrightarrow I(1) \longrightarrow I(2) \longrightarrow 0$.
(iii) Prove that $\tau M \cong M$.

4. Prove that whenever the Coxeter matrix of A is defined, we have that $\Phi_A(\underline{\dim} P) = -\underline{\dim}\, \nu P$ for every projective A-module.

5. Let Q be the quiver $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$.

(i) Show that $\tau^j S(1) = S(j+1)$ for $j < n$.
(ii) When $n = 3$, show that any indecomposable module is either simple, injective or projective.
(iii) When $n = 4$, show that there exists an indecomposable non-simple, non-injective and non-projective module of the form τM.